

Arcs in the Plane*

Joan E. Hart† and Kenneth Kunen‡§

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Abstract

Assuming PFA, every uncountable subset $E$ of the plane meets some $C^1$ arc in an uncountable set. This is not provable from MA($\aleph_1$), although in the case that $E$ is analytic, this is a ZFC result. The result is false in ZFC for $C^2$ arcs, and the counter-example is a perfect set.

1 Introduction

As usual, an arc in $\mathbb{R}^n$ is a set homeomorphic to a closed bounded subinterval of $\mathbb{R}$. A (simple) path is a homeomorphism $g$ mapping a compact interval onto $A$. For $k \geq 1$, a path is $C^k$ iff it is a $C^k$ function, and an arc $A$ is $C^k$ iff $A$ is the image of some $C^k$ path $g$, with $g'(t) \neq 0$ for all $t$; equivalently, $A$ has a $C^k$ arc length parameterization. Also, $A$ is $C^\infty$ iff it is $C^k$ for all $k$. We consider the following:

*Question.* For $n \geq 2$, if $E \subseteq \mathbb{R}^n$ is uncountable, must there be a “nice” arc $A$ such that $E \cap A$ is uncountable?

Obviously, the answer will depend on the definition of “nice”. We should expect ZFC results for closed $E$ (equivalently, for analytic $E$), and independence results for arbitrary $E$. In general, under CH things are as bad as possible, and under PFA, things are as good as possible. In most cases, the results are the same for all $n \geq 2$, and trivial for $n = 1$.

For arbitrary arcs, the results are quite old. In ZFC, every closed uncountable set meets some arc in an uncountable set. For $n \geq 2$, arcs are nowhere dense in $\mathbb{R}^n$; so under CH there is a Luzin set that meets every arc in a countable set. At

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†University of Wisconsin, Oshkosh, WI 54901, U.S.A., hartj@uwosh.edu
‡University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu
§Both authors partially supported by NSF Grant DMS-0456653.
the other extreme, under $\text{MA}(\aleph_1)$, every uncountable $E \subseteq \mathbb{R}^n$ meets some arc in an uncountable set.

If “nice” means “straight line”, then there is a trivial counter-example: a perfect set $E$ which meets every line in at most two points.

Paper [3] introduces results where “nice” means “almost straight”:

**Definition 1.1** Let $\rho : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ be the perpendicular retraction given by $\rho(x) = x/\|x\|$. Then $A \subseteq \mathbb{R}^n$ is $\varepsilon$-directed iff for some $v \in S^{n-1}$, $\|\rho(x-y) - v\| \leq \varepsilon$ or $\|\rho(x-y) + v\| \leq \varepsilon$ whenever $x, y$ are distinct points of $A$.

The retraction $\rho(x-y)$ may be viewed as the direction from $y$ to $x$. Every $A \subseteq \mathbb{R}^n$ is trivially $\sqrt{2}$-directed, and $A$ is 0-directed iff $A$ is contained in a straight line. If “nice” means “$\varepsilon$-directed”, a counter-example to the Question is consistent with $\text{MA}(\aleph_1)$. By [3], the existence of a weakly Luzin set is consistent with $\text{MA}(\aleph_1)$, and whenever $\varepsilon < \sqrt{2}$, a weakly Luzin set (see [3] Definition 2.4) meets every $\varepsilon$-directed set in a countable set. However, under $\text{SOCA}$, which follows from PFA, whenever $\varepsilon > 0$, every uncountable set meets some $\varepsilon$-directed arc in an uncountable set (see Lemma 4.1). Every $C^1$ arc is a finite union of $\varepsilon$-directed arcs, and hence we get the stronger:

**Theorem 1.2** PFA implies that every uncountable subset of $\mathbb{R}^n$ meets some $C^1$ arc in an uncountable set.

$\text{MA}(\aleph_1)$ is not sufficient for this theorem, because, as in the $\varepsilon$-directed case ($\varepsilon < \sqrt{2}$), a weakly Luzin set provides a counter-example. Theorem 1.2 and the following ZFC theorem for closed sets are proved in Section 4.

**Theorem 1.3** If $P \subseteq \mathbb{R}^n$ is closed and uncountable, then there is a $C^1$ arc $A$ with a Cantor set $Q \subseteq P \cap A$. Hence, for every $\varepsilon > 0$, $P$ meets some $\varepsilon$-directed arc in an uncountable set.

If the Question asks for a $C^2$ arc, then a ZFC counter-example exists in the plane, and hence in any $\mathbb{R}^n$ ($n \geq 2$). The counter-example, given in Theorem 1.5, is a non-squiggly subset of the plane. A simple example of a non-squiggly set is a $C^1$ arc whose tangent vector either always rotates clockwise or always rotates counter-clockwise. In particular, such an arc may be the graph of a convex function $f \in C^1([0, 1], \mathbb{R})$; a real differentiable function is convex iff its derivative is a monotonically increasing function. But non-squiggly makes sense for non-smooth arcs, and in fact for arbitrary subsets of the plane:

**Definition 1.4** A $\subseteq \mathbb{R}^2$ is non-squiggly iff there is a $\delta$, with $0 < \delta \leq \infty$, such that whenever $\{x, y, z, t\} \in [A]^4$ and $\text{diam}(\{x, y, z, t\}) \leq \delta$, point $t$ is not interior to triangle $xyz$. 

Theorem 1.5 There is a perfect non-squiggly set $P \subseteq \mathbb{R}^2$ which lies in a $C^1$ arc $A$ and which meets each $C^2$ arc in a finite set. Moreover, the $C^1$ arc $A$ may be taken to be the graph of a convex function.

As “nice” notions, non-squiggly is orthogonal to smooth:

Theorem 1.6 There is a perfect set $P \subseteq \mathbb{R}^2$ which lies in a $C^\infty$ arc and which meets every non-squiggly set in a countable set.

Note that by Ramsey’s Theorem, every infinite set in $\mathbb{R}^2$ has an infinite non-squiggly subset.

In Definition 1.4, allowing $\delta < \infty$ makes non-squiggly a local notion; so, piecewise linear arcs and some spirals (such as $r = \theta$; $0 \leq \theta < \infty$) are non-squiggly. However, the results of this paper would be unchanged if we simply required $\delta = \infty$. For $0 < \delta \leq \infty$, if $E \subseteq \mathbb{R}^2$ meets a non-squiggly set $A$ in an uncountable set, then $E$ has uncountable intersection with a subset of $A$ whose diameter is at most $\delta$.

The proof of Theorem 1.5 uses the assumption that each $C^2$ arc is parameterized by some $g$ whose derivative is nowhere 0. Dropping this requirement on $g'$ yields a weaker notion of $C^\infty$, and a different result. Call a $C^k$ arc strongly $C^k$, and say that an arc is weakly $C^k$ iff it is the image of a $C^k$ path. Then, an arc is weakly $C^\infty$ iff it is weakly $C^k$ for all $k$.

Theorem 1.7 If $E \subseteq \mathbb{R}^n$ is bounded and infinite, then it meets some weakly $C^\infty$ arc in an infinite set.

Theorems 1.5 and 1.6 are proved in Section 5; Theorem 1.7 and some related facts are proved in Section 6.

2 Remarks on Hermite Splines

We construct the arc of Theorem 1.3 by first producing a “nice” Cantor set $Q \subseteq P$. Then we apply results, described in this section, that make it possible to draw a smooth curve through a closed set. These results are a natural extension of results of Hermite for drawing a curve through a finite set. Our proof of Theorem 1.3 reduces the problem to the case where $Q \subseteq \mathbb{R}^2$ is the graph of a function with domain $D \subseteq \mathbb{R}$; then we extend this function to all of $\mathbb{R}$ to produce the desired arc.

First consider the case $|D| = 2$, or interpolation on an interval $[a_1, a_2]$; we find $f \in C^1(\mathbb{R})$ with predetermined values $b_1, b_2$ and slopes $s_1, s_2$ at $a_1, a_2$, and we bound $f, f'$ on $[a_1, a_2]$ in terms of the three slopes: $s := (b_2 - b_1)/(a_2 - a_1)$, and $s_1, s_2$. Following Hermite, $f$ will be the natural cubic interpolation function. Our bounds show that if $s, s_1, s_2$ are all close to each other, then $f$ is close to the linear interpolation function $L$. 
Lemma 2.1 Given $s_1, s_2, b_1, b_2$ and $a_1 < a_2$, let $s = (b_2 - b_1)/(a_2 - a_1)$, and let $L(x) = b_1 + s(x - a_1)$. Let $M = \max(|s_1 - s|, |s_2 - s|)$. Then there is a cubic $f$ with each $f(a_i) = b_i$ and each $f'(a_i) = s_i$, such that

1. $|(f(x_2) - f(x_1))/(x_2 - x_1) - s| \leq 3M$ whenever $a_1 \leq x_1 < x_2 \leq a_2$.

Moreover, for all $x \in [a_1, a_2]$:

2. $|f'(x) - s| \leq 3M$.

3. $|f(x) - L(x)| \leq 2M(a_2 - a_1)$.

Proof. (1) follows from (2) and the Mean Value Theorem. Now, let

$$f(x) = L(x) + \beta_2(x - a_1)^2(x - a_2) + \beta_1(x - a_1)(x - a_2)^2$$

$$f'(x) = s + \beta_2(x - a_1)^2 + \beta_1(x - a_2)^2 + 2(\beta_2 + \beta_1)(x - a_1)(x - a_2).$$

Then $f(a_i) = b_i$ is obvious, and setting $\beta_i = (s_i - s)/(a_2 - a_1)^2$ we get $f'(a_i) = s_i$.

To see (2) and (3), note that $|\beta_i| \leq M/(a_2 - a_1)^2$, and $(x-a_1)(a_2-x) \leq (a_2-a_1)^2/4$ (the maximum of $(x-a_1)(a_2-x)$ occurs at the midpoint $x = \frac{a_1+a_2}{2}$).

Next, we consider extending, to all of $\mathbb{R}$, a $C^1$ function defined on a closed $D \subset \mathbb{R}$. First note that there are two possible meanings for “$f \in C^1(D)$”:

Definition 2.2 Assume that $f, h \in C(D, \mathbb{R})$, where $D$ is a closed subset of $\mathbb{R}$. Then $f' = h$ in the strong sense iff

$$\forall x \in D \forall \varepsilon > 0 \exists \delta > 0 \forall x_1, x_2 \in D \left[ x_1 \neq x_2 \& |x_1 - x|, |x_2 - x| < \delta \Rightarrow \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} - h(x) \right| < \varepsilon \right].$$

The usual or weak sense would only require this with $x_1$ replaced by the point $x$. When $D$ is an interval, the two senses are equivalent by the continuity of $h$ and the Mean Value Theorem. Note that $f' = h$ in the strong sense iff there is a $g \in C(D \times D, \mathbb{R})$ such that $g(x, x) = h(x)$ for each $x$ and $g(x_1, x_2) = g(x_2, x_1) = (f(x_2) - f(x_1))/(x_2 - x_1)$ whenever $x_1 \neq x_2$.

If $D$ is finite, then $f' = h$ in the strong sense for any $f, h : D \rightarrow \mathbb{R}$, and the cubic Hermite spline is an $\tilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ with $\tilde{f}|D = f$ and $\tilde{f}'|D = h$. The following lemma generalizes this to an arbitrary closed $D$:

Lemma 2.3 Assume that $f, h \in C(D, \mathbb{R})$, where $D$ is a closed subset of $\mathbb{R}$, and $f'' = h$ in the strong sense. Then there are $\tilde{f}, \tilde{h} \in C(\mathbb{R}, \mathbb{R})$ such that $\tilde{f}' = h$, $\tilde{f} \supseteq f$, and $\tilde{h} \supseteq h$. 
Proof. Let $J$ be the collection of pairwise disjoint open intervals covering $\mathbb{R} \setminus D$. For each interval $J \in J$, we shall define $\tilde{f}, \tilde{h}$ on $J$. If $J$ is the unbounded interval $(a_1, \infty)$, with $a_1 \in D$, define $\tilde{f}$ and $\tilde{h}$ by the linear 
$f(x) = f(a_1) + (x - a_1)h(a_1)$ and $\tilde{h}(x) = h(a_1)$, for $x \in J$. Then $\tilde{f}, \tilde{h}$ are continuous on $J$ and $\tilde{f}' = \tilde{h}$ on $J$. At $a_1$, the derivative of $\tilde{f}$ from the right is $h(a_1)$; the derivative of $\tilde{f}$ from the left, as well as the continuity of $\tilde{f}, \tilde{h}$ from the left, depend on how we extend $f$ to the bounded intervals.

The unbounded interval $(-\infty, a_2)$ is handled likewise.

Say $J = (a_1, a_2)$, with $a_1, a_2 \in D$. On $J$, let $\tilde{f}$ be the cubic obtained from Lemma 2.1, with $b_i = f(a_i)$ and $s_i = h(a_i)$. Then $\tilde{h}$ is the quadratic $\tilde{f}'$ on $J$.

To finish, we verify that $\tilde{f}, \tilde{h}$ are continuous and $\tilde{f}' = \tilde{h}$ on $\mathbb{R}$. Fix $z \in D$. Since differentiability implies continuity, it suffices to show that $\tilde{h}$ is continuous at $z$, and that $h(z) = \tilde{f}'(z) = \lim_{x \to z} (\tilde{f}(x) - \tilde{f}(z))/(x - z)$. We verify the continuity of $\tilde{h}$ from the left at $z$, and the difference quotient’s limit for $x$ approaching $z$ from the left; a similar argument handles these from the right. Let $\sigma = h(z) = \tilde{h}(z)$. Fix $\varepsilon > 0$. Apply continuity of $f, h$ on $D$, and the fact that $f' = h$ in the strong sense, to fix $\delta > 0$ such that whenever $z - \delta < a_1 < a_2 < z$ with $a_1, a_2 \in D$, the quantities $|s - \sigma|, |s_i - \sigma|, |b_i - f(z)|, |(f(a_2) - f(z))/ (a_2 - z) - \sigma|$ are all less than $\varepsilon$, where $s_i = h(a_i)$ and $b_i = f(a_i)$, for $i = 1, 2$, and $s = (b_2 - b_1)/(a_2 - a_1)$. Let $M = \max(|s_1 - s|, |s_2 - s|)$, as in Lemma 2.1; so $M \leq 2\varepsilon$.

Assume that $z$ is a limit from the left of points of $D$ and of points of $\mathbb{R} \setminus D$; otherwise checking continuity and the derivative from the left is trivial. Thus, $\delta$ may be taken small enough so that $(z - \delta, z)$ misses any unbounded interval in $J$. For $a_1, a_2 \in D$ with $(a_1, a_2) \in J$ and $x \in \mathbb{R}$ with $z - \delta < a_1 \leq x < a_2 < z$, the bounds from Lemma 2.1 imply that $|\tilde{h}(x) - \sigma| \leq |\tilde{h}(x) - s| + |s - \sigma| \leq 3M + \varepsilon \leq 7\varepsilon$.

So $\tilde{h}$ is continuous. To see that $h(z) = \tilde{f}'(z)$, observe that by elementary geometry, the slope $(\tilde{f}(x) - \tilde{f}(z))/(x - z)$ is between the slopes $(\tilde{f}(x) - \tilde{f}(a_2))/(x - a_2)$ and $(\tilde{f}(a_2) - \tilde{f}(z))/(a_2 - z)$. Applying Lemma 2.1 again, $|(\tilde{f}(x) - \tilde{f}(a_2))/(x - a_2) - \sigma| \leq 3M + \varepsilon \leq 7\varepsilon$, so we are done. 

\section{Some Flavors of OCA}

The proofs of Theorems 1.2 and 1.3 will require the results of this section.

\begin{definition}
For any set $E$, let $E^\dagger = (E \times E) \setminus \{(x, x) : x \in E\}$. If $W \subseteq E^\dagger$ with $W = W^{-1}$, then $T \subseteq E$ is $W$-free iff $T^\dagger \cap W = \emptyset$, and $T$ is $W$-connected iff $T^\dagger \subseteq W$.

Then SOCA is the assertion that whenever $E$ is an uncountable separable metric space and $W = W^{-1} \subseteq E^\dagger$ is open, there is either an uncountable $W$-free set or an uncountable $W$-connected set.
\end{definition}
SOCA follows from PFA, but not from MA(\(\aleph_1\)). It clearly contradicts CH. However, it is well-known [2] that SOCA is a ZFC theorem when \(E\) is Polish:

**Lemma 3.2** Assume that \(E\) is an uncountable Polish space, \(W \subseteq E^\dagger\) is open, and \(W = W^{-1}\). Then there is a Cantor set \(Q \subseteq E\) which is either \(W\)-free or \(W\)-connected.

**Proof.** Shrinking \(E\), we may assume that \(E\) is a Cantor set; in particular, non-empty open sets are uncountable. Assume that no Cantor subset is \(W\)-free. Since \(W\) is open, the closure of a \(W\)-free set is \(W\)-free; thus every \(W\)-free set has countable closure, and is hence nowhere dense.

Now, inductively construct a tree, \(\{P_s : s \in 2^{<\omega}\}\). Each \(P_s\) is a non-empty clopen subset of \(E\), with \(\text{diam}(P_s) \leq 2^{-\text{lh}(s)}\). \(P_s^{-0}\) and \(P_s^{-1}\) are disjoint subsets of \(P_s\) such that \((P_s^{-0} \times P_s^{-1}) \subseteq W\). Let \(Q = \bigcup\{\bigcap_n P_{f|n} : f \in 2^\omega\}\); then \(Q\) is \(W\)-connected.

An “open covering” version of SOCA follows by induction on \(\ell\):

**Lemma 3.3** Let \(E\) be an uncountable separable metric space, with \(E^\dagger = \bigcup_{i<\ell} W_i\), where \(\ell \in \omega\) and each \(W_i = W_i^{-1}\) is open in \(E^\dagger\). Assuming SOCA, there is an uncountable \(T \subseteq E\) such that \(T\) is \(W_i\)-connected for some \(i\). In the case that \(E\) is Polish, this is a ZFC result and \(T\) can be made perfect.

There is also a version of this lemma obtained by replacing the covering by a continuous function:

**Lemma 3.4** Assume that \(E\) is an uncountable Polish space, \(F\) is a compact metric space, \(g \in C(E^\dagger, F)\), and \(g(x, y) = g(y, x)\) whenever \(x \neq y\). Then there is a Cantor set \(Q \subseteq E\) such that \(g|Q^\dagger\) extends continuously to some \(\hat{g} \in C(Q \times Q, F)\).

**Proof.** Construct a tree, \(\{P_s : s \in 2^{<\omega}\}\). Each \(P_s\) is a Cantor subset of \(E\), with \(\text{diam}(P_s) \leq 2^{-\text{lh}(s)}\). \(P_s^{-0}\) and \(P_s^{-1}\) are disjoint subsets of \(P_s\). Also, apply Lemma 3.3 to get \(\text{diam}(g(P_s^\dagger)) \leq 2^{-\text{lh}(s)}\). Let \(Q = \bigcup\{\bigcap_n P_{f|n} : f \in 2^\omega\}\).

Now, to prove Theorem 1.2, we need, under PFA, a version of Lemma 3.4 where \(E\) is just an uncountable subset of a Polish space. We begin with the following, from Abraham, Rubin, and Shelah [1]:

**Theorem 3.5** Assume PFA. Then OCA[ARS] holds. That is, let \(E\) be a separable metric space of size \(\aleph_1\). Assume that \(E^\dagger = \bigcup_{i<\ell} W_i\), where \(\ell \in \omega\) and each \(W_i = W_i^{-1}\) is open in \(E^\dagger\). Then \(E\) can be partitioned into sets \(\{A_j : j \in \omega\}\) such that for each \(j\), \(A_j\) is \(W_i\)-connected for some \(i\).
3 SOME FLAVORS OF OCA

The terminology OCA\textsubscript{ARS} was used by Moore [4] to distinguish it from other flavors of the Open Coloring Axiom in the literature. Actually, [1] does not mention PFA, but rather its Theorem 3.1 shows, by iterated ccc forcing, that OCA\textsubscript{ARS} is consistent with MA($\aleph_1$); but the same proof shows that it is true under PFA. In our proof of Theorem 1.2, we only need MA($\aleph_1$) plus OCA\textsubscript{ARS}, so in fact every model of $2^{\aleph_0} = \aleph_1 \land 2^{\aleph_1} = \aleph_2$ has a ccc extension satisfying the result of Theorem 1.2.

To use OCA\textsubscript{ARS} for our version of Lemma 3.4, we need the $A_j$ of Theorem 3.5 to be clopen. This is not always possible, but can be achieved if we shrink $E$:

**Lemma 3.6** Assume MA($\aleph_1$). Assume that $X$ is a Polish space and $E \in [X]^{\aleph_1}$. For each $n \in \omega$, let $\{A^n_j : j \in \omega\}$ partition $E$ into $\aleph_0$ sets. Then there is a Cantor set $Q \subseteq X$ and, for each $n$, a partition of $Q$ into disjoint relatively clopen sets $\{K^n_j : j \in \omega\}$ such that $|Q \cap E| = \aleph_1$ and each $K^n_j \cap E = A^n_j \cap Q$.

**Proof.** Note that for each $n$, compactness of $Q$ implies that all but finitely many of the $K^n_j$ will be empty.

For $s \in \omega^{<\omega}$, let $A_s = \bigcap \{A^n_s(n) : n < \text{lh}(s)\}$, with $A_\emptyset = E$. Shrinking $E, X$, we may assume that whenever $U \subseteq X$ is open and non-empty, $|E \cap U| = \aleph_1$ and each $|A_s \cap U|$ is either 0 or $\aleph_1$.

Let $\mathcal{B}$ be a countable open base for $X$, with $X \in \mathcal{B}$. Call $\mathcal{T}$ a nice tree iff:

1. $\mathcal{T}$ is a non-empty subset of $\mathcal{B}\setminus\{\emptyset\}$ which is a tree under the order $\subseteq$, with root node $X$.
2. $\mathcal{T}$ has height $\text{ht}(\mathcal{T})$, where $1 \leq \text{ht}(\mathcal{T}) \leq \omega$.
3. If $U \in \mathcal{T}$ is at level $\ell$ with $\ell + 1 < \text{ht}(\mathcal{T})$, then $U$ has finitely many but at least two children in $\mathcal{T}$, and the closures of the children are pairwise disjoint and contained in $U$.
4. If $U \in \mathcal{T}$ is at level $\ell > 0$, then $\text{diam}(U) \leq 1/\ell$.

This labels the levels $0, 1, 2, \ldots$, with $\text{ht}(\mathcal{T})$ the first empty level. Let $L_\ell(\mathcal{T})$ be the set of nodes at level $\ell$. By (1)–(3), each $L_\ell(\mathcal{T})$ is a finite pairwise disjoint collection. When $\text{ht}(\mathcal{T}) = \omega$, let $Q_\mathcal{T} = \bigcap_{\ell \in \omega} L_\ell(\mathcal{T}) = \bigcap_{\ell \in \omega} \text{cl}(\bigcup L_\ell(\mathcal{T}))$. Then $Q_\mathcal{T}$ is a Cantor set, so it is natural to force with finite trees approximating $\mathcal{T}$. Since many Cantor sets are disjoint from $E$, each forcing condition $p$ will have, as a side condition, a finite $I_p \subseteq E$ which is forced to be a subset of $Q$.

Define $p \in \mathbb{P}$ iff $p$ is a triple $(\mathcal{T}, I, \varphi) = (\mathcal{T}_p, I_p, \varphi_p)$, such that:

a. $\mathcal{T}$ is a nice tree of some finite height $h = h_p \geq 1$.

b. $I$ is finite and $I \subseteq E \cap \bigcup L_{h-1}(\mathcal{T})$.

c. $\varphi : \mathcal{T} \rightarrow \omega^{<\omega}$ with $\varphi(U) \in \omega$ for $U \in L_\ell(\mathcal{T})$.

d. $\varphi(V) \supseteq \varphi(U)$ whenever $V \subseteq U$. 


e. If \( s = \varphi(U) \) then \( A_s \cap U \neq \emptyset \) and \( I_p \subseteq A_s \).

Define \( q \leq p \) iff \( T_q \) is an end extension of \( T_p \) and \( I_q \supseteq I_p \) and \( \varphi_q \supseteq \varphi_p \). Then \( \mathbb{1} = (\{X\}, \emptyset, \{(X, \emptyset)\}) \). \( \mathbb{P} \) is ccc (and \( \sigma \)-centered) because \( p, q \) are compatible whenever \( T_p = T_q \) and \( \varphi_p = \varphi_q \). If \( G \) is a filter meeting the dense sets \( \{p : h_p > n\} \) for each \( n \), then \( G \) defines a tree \( T = T_G = \bigcup\{T_p : p \in G\} \) of height \( \omega \), and \( Q = Q_T \) is a Cantor set. We also have \( \varphi_G = \bigcup\{\varphi_p : p \in G\} \), so \( \varphi_G : T_G \to \omega^\omega \); also, let \( I_G = \bigcup\{I_p : p \in G\} \).

Note that for each \( x \in E \), \( \{p : x \in I_p \vee x \notin \bigcup L_{h_p-1}(T_p)\} \) is dense in \( \mathbb{P} \). If \( G \) meets all these dense sets, then \( Q \cap E = I_G \). We may then let \( K_j^q = Q \cap \bigcup\{U \in L_{n+1}(T_G) : \varphi(U)(n) = j\} \).

Finally, if we list \( E \) as \( \{e_\beta : \beta < \omega_1\} \), note that each set \( \{p : \exists \beta > \alpha [e_\beta \in I_p]\} \) is dense, so that we may force \( Q \cap E \) to be uncountable. 

**Lemma 3.7** Assume PFA. Assume that \( X \) is a Polish space, \( F \) is a compact metric space, \( E \in [X]^{\aleph_1} \), \( g \in C(X^+, F) \), and \( g(x, y) = g(y, x) \) whenever \( x \neq y \). Then there is a Cantor set \( Q \subseteq X \) such that \( |Q \cap E| = \aleph_1 \) and \( g|Q^+ \) extends continuously to some \( \hat{g} \in C(Q \times Q, F) \).

**Proof.** For each \( n \), we may use compactness of \( F \) to cover \( X^+ \) by finitely many open sets, \( W_i^n = (W_i^n)^{-1} \) for \( i < \ell_n \), such that each \( \text{diam}(g(W_i^n)) \leq 2^{-n} \). It follows by Theorem 3.5 that for each \( n \), we may partition \( E \) into sets \( \{A_j^n : j \in \omega\} \) such that each \( A_j^n \) is \( W_i^n \)-connected for some \( i \), so that \( \text{diam}(g((A_j^n)^+)) \leq 2^{-n} \).

By Lemma 3.6, we have a Cantor set \( Q \subseteq X \) and, for each \( n \), a partition of \( Q \) into disjoint relatively clopen sets \( \{K_j^n : j \in \omega\} \) such that \( |Q \cap E| = \aleph_1 \) and each \( K_j^n \cap E = A_j^n \cap Q \). Shrinking \( Q \), we may assume \( Q \cap E \) is dense in \( Q \), so that each \( A_j^n \cap Q \) is dense in \( K_j^n \) and \( \text{diam}(g((K_j^n)^+)) \leq 2^{-n} \).

Now, fix \( x \in Q \). For each \( n \), \( x \) lies in exactly one of the \( K_j^n \), and we may let \( H^n = \text{cl}(g((K_j^n)^+)) \) for that \( j \). Then \( \bigcap_n H^n \) is a singleton, and we may define \( \hat{g} \) on the diagonal by \( \{\hat{g}(x, x)\} = \bigcap_n H^n \). It is easily seen that this \( \hat{g} \) is continuous on \( Q \times Q \).

### 4 Proofs of Positive Results

**Lemma 4.1** Fix an uncountable \( E \subseteq \mathbb{R}^n \) and an \( \varepsilon > 0 \). Assuming SOCA, there is an uncountable \( T \subseteq E \) such that \( T \) is \( \varepsilon \)-directed. In the case that \( E \) is Polish, this is a ZFC result and \( T \) can be made perfect.
4 PROOFS OF POSITIVE RESULTS

Proof. Let \( \{V_i : i < \ell\} \) be an open cover of \( S^{n-1} \) by sets of diameter less than \( \varepsilon \), and apply Lemma 3.3 with \( W_i = \{(x, y) \in E^\ell : \rho(x - y) \in V_i\} \).  

Proof of Theorem 1.3. Applying Lemma 4.1 and shrinking \( P \), we may assume that \( P \) is a Cantor set and that \( P \) is \( 2 \sin(22.5^\circ) \)-directed; so, the direction between any two points of \( P \) is within \( 45^\circ \) of some fixed direction. Rotating coordinates, we may assume that this fixed direction is along the \( x \)-axis, where we label our \( n \) axes as \( x, y^1, \ldots, y^{n-1} \). Now, \( P \) is (the graph of) a function which expresses \( (y^1, \ldots, y^{n-1}) \) as a function of \( x \), and \( D := \text{dom}(P) \) is a Cantor set. Write \( P(x) \) as \( (P^1(x), \ldots, P^{n-1}(x)) \).

The \( xy^i \)-planar slopes of \( P \) are all in \([-1, 1] \). That is, for \( x_1, x_2 \in D \) with \( x_1 \neq x_2 \), let \( g^i(x_1, x_2) = (P^i(x_2) - P^i(x_1))/(x_2 - x_1) \); then \( |g^i(x_1, x_2)| \leq 1 \) for all \( x_1, x_2 \). Each \( g^i \in C(D^i, [0, 1]) \) and \( g^i(x_1, x_2) = g^i(x_2, x_1) \) whenever \( x_1 \neq x_2 \). Applying Lemma 3.4 with \( F = [0, 1]^{n-1} \) and shrinking \( D \) if necessary, we may assume that each \( g^i \) extends continuously to some \( \hat{g}^i \in C(D \times D, [0, 1]) \). Let \( h^i(x) = \hat{g}^i(x, x) \). Then \( h^i \) is the derivative of \( P^i \) in the strong sense. Now, we may apply Lemma 2.3 on each coordinate separately to obtain a \( C^1 \) arc \( A \supseteq P \); \( A \) is the graph of a \( C^1 \) function \( x \mapsto (A^1(x), \ldots, A^{n-1}(x)) \) defined on an interval containing \( D \).

Proof of Theorem 1.2. Given Lemma 3.7, the proof is almost identical to the proof of Theorem 1.3.

When \( E \subseteq \mathbb{R}^n \) has size exactly \( \aleph_1 \), and the Question of Section 1 has a positive answer, it is natural to ask whether \( E \) can be covered by \( \aleph_0 \) “nice” arcs. For example, under \( \text{MA}(\aleph_1) \), \( E \) is covered by \( \aleph_0 \) Cantor sets, and hence by \( \aleph_0 \) arcs. One can also improve Theorem 1.2:

Theorem 4.2 PFA implies every \( E \subseteq \mathbb{R}^n \) of size \( \aleph_1 \) can be covered by \( \aleph_0 \) \( C^1 \) arcs.

The proof mimics the proof of Theorem 1.2, but uses improved versions of Lemmas 4.1, 3.6 and 3.7. The new and improved Lemma 4.1 gets \( E \) covered by \( \aleph_0 \) \( \varepsilon \)-directed sets, using Theorem 3.5 rather than SOCA.

The covering versions of Lemmas 3.6 and 3.7 get Cantor sets \( Q_\ell \subseteq X \) for \( \ell \in \omega \) satisfying the conditions of the lemmas and so that \( E \subseteq \bigcup_\ell Q_\ell \). To get the \( Q_\ell \) for \( \ell \in \omega \), force with the finite support product of \( \omega \) copies of the poset \( \mathbb{P} \) described in the proof of Lemma 3.6. Then, use the \( Q_\ell \) to prove the covering version of Lemma 3.7. Even though the proof of Lemma 3.7 shrinks \( Q \), it does so by deleting at most countably many points from \( E \), so these points may be covered by \( \aleph_0 \) straight lines. Thus, \( E \) will be covered by \( \bigcup_\ell Q_\ell \) together with a countable union of lines.
5 Proofs of Negative Results

Lemma 5.1 Let $D \subset \mathbb{R}$ be closed. Then there is an $h \in C^\infty(\mathbb{R})$ such that $h(x) \geq 0$ for all $x$ and $D = \{x \in \mathbb{R} : h(x) = 0\}$.

Proof. Let $U = \mathbb{R} \setminus D$; we shall call our function $h_U$. If $U = (a, b)$, then such $h_U$ are in standard texts; for example, let $h_{(a,b)}(x) = \exp(-1/(x-a)(b-x))$ for $x \in (a,b)$ and $0$ otherwise. Now, say $U = \bigcup_{n \in \omega} J_n$, where each $J_n$ is a bounded open interval. Let $h_U = \sum_{n \in \omega} c_n h_{J_n}$, where each $c_n > 0$ and the $c_n$ are small enough so that for each $\ell \in \omega$, the $\ell$th derivative $h_U^{(\ell)}$ is the uniform limit of the sum $\sum_{n \in \omega} c_n h_{J_n}^{(\ell)}$. 

Proof of Theorem 1.6. Let $D \subset \mathbb{R}$ be a Cantor set. Integrating the function of Lemma 5.1, fix $f \in C^\infty(\mathbb{R})$ such that $f'(x) \geq 0$ for all $x$ and $D = \{x \in \mathbb{R} : f'(x) = 0\}$. Then $f$ is strictly increasing.

Let $P$ be the graph of $f|D$. Fix an uncountable $A \subseteq P$, and assume that $A$ is non-squiggly; we shall derive a contradiction. Fix $\delta > 0$ as in Definition 1.4; then, shrinking $A$, we may assume that diam$(A) \leq \delta$ so that whenever $\{x, y, z, t\} \in [A]^4$, point $t$ is not interior to triangle $xyz$.

Let $S$ be an infinite subset of dom$(A)$ such that every point of $S$ is a limit, from the left and right, of other points of $S$.

Now, fix $a, b, c \in S$ with $a < b < c$; then $f(a) < f(b) < f(c)$. Let $L$ be the straight line passing through $(a, f(a))$ and $(c, f(c))$. Moving $b$ slightly if necessary, we may assume (since $f'(b) = 0$) that $L$ does not pass through $(b, f(b))$. Then either $L(b) > f(b)$ or $L(b) < f(b)$.

Suppose that $L(b) > f(b)$. Consider triangle $(a, f(a)), (b, f(b)), (c, f(c))$. One leg of this triangle is the graph of $L|[a,c]$, which passes above the point $(b, f(b))$. Since all three legs have positive slope and $f'(b) = 0$, the points $(b-\varepsilon, f(b-\varepsilon))$ are interior to the triangle when $\varepsilon > 0$ is small enough. Choosing such $\varepsilon$ with $b-\varepsilon \in S$ yields a contradiction.

$L(b) < f(b)$ is likewise contradictory, using points $(b + \varepsilon, f(b + \varepsilon))$. 

Observe that the arc in Theorem 1.6 cannot be real-analytic, since if $f : [0, 1] \to \mathbb{R}$ is real-analytic, then $[0, 1]$ can be decomposed into finitely many intervals on which either $f'' \geq 0$ or $f'' \leq 0$. On each of these intervals, the graph of $f$ is non-squiggly.

Proof of Theorem 1.5. As in the proof of Theorem 1.6, let $D \subset \mathbb{R}$ be a Cantor set, and fix $f \in C^\infty(\mathbb{R})$ such that $f$ is strictly increasing, $f'(y) \geq 0$ for all $y$, and $D = \{y \in \mathbb{R} : f'(y) = 0\}$. Also, to simplify notation, assume that $f(\mathbb{R}) = \mathbb{R}$, so that $\varphi := f^{-1} \in C(\mathbb{R})$ and is also a strictly increasing function. Let $K = f(D)$; so $K$ is also a Cantor set. Then $\varphi$ is $C^\infty$ on $\mathbb{R} \setminus K$, and $\varphi'(x) = +\infty$ for $x \in K$. Integrating, fix $\psi \in C^1(\mathbb{R})$ such that $\psi' = \varphi$; so $\psi$ is a convex function.
Note that whenever \( x \in K \) and \( M > 0 \), there is an \( \varepsilon > 0 \) such that \( \varphi'(u) \geq M \) whenever \( |u - x| < \varepsilon \). When \( x - \varepsilon < a \leq v \leq b < x + \varepsilon \), we can integrate this to get \( \varphi(a) + M(v - a) \leq \varphi(v) \leq \varphi(b) - M(b - v) \). Integrating again yields
\[
(b - a)\varphi(a) + (b - a)^2M/2 \leq \psi(b) - \psi(a) \leq (b - a)\varphi(b) - (b - a)^2M/2
\]
This implies that, for \( x \in K \),
\[
\lim_{t \to 0} \frac{(\psi(x + t) - \psi(x))/t - \varphi(x)}{t} = +\infty \quad \text{(*)}
\]
the argument can be broken into two cases: \( t \searrow 0 \) (consider \( a = x < x + t = b \)) and \( t \nearrow 0 \) (consider \( a = x + t < x = b \)).

Now let \( P = \psi|K \); so \( P \) is a Cantor set in \( \mathbb{R}^2 \). Suppose that \( P \) meets the \( C^2 \) arc \( A \) in an infinite set. Since the intersection is compact, it contains a limit point \((x_0, y_0)\). At \((x_0, y_0)\), the tangent to the arc \( A \) is parallel to the tangent of the \( C^1 \) arc \( y = \psi(x) \); in particular, this tangent is not vertical. Thus, replacing \( A \) by a segment thereof, we may assume that \( A \) is the arc \( y = \xi(x) \), where \( \xi \) is a \( C^2 \) function defined in some neighborhood of \( x_0 \). Now \( y_0 = \xi(x_0) = \psi(x_0) \) and \( \xi'(x_0) = \psi'(x_0) = \varphi(x_0) \). Also, since \((x_0, y_0)\) is a limit point of the intersection, there are non-zero \( t_k \), for \( k \in \omega \), converging to 0, such that each \( \psi(x_0 + t_k) = \xi(x_0 + t_k) \). Applying Taylor’s Theorem to \( \xi \),
\[
\psi(x_0 + t_k) = \psi(x_0) + \varphi(x_0)t_k + \frac{1}{2}\xi''(z_k)t_k^2
\]
for some \( z_k \) between \( x_0 \) and \( x_0 + t_k \).
Since \( \xi''(z_k) \to \xi''(x_0) \), we have
\[
[(\psi(x_0 + t) - \psi(x_0))/t - \varphi(x_0)]/t_k \to \xi''(x_0)/2 \quad ,
\]
contradicting (*). \( \square \)

If \( \psi \) were \( C^2 \), the limit in (*) would be \( \psi''(x)/2 \neq \infty \) (by Taylor’s Theorem). Moreover, the Cantor set \( P = \psi|K \) meets any \( C^2 \) arc in a finite set. This illustrates a difference between \( C^1 \) and \( C^2 \): rotation can cure an infinite derivative, but not an infinite second derivative. Even though \( \varphi'(x) = \infty \) for \( x \in K \), rotating the graph of \( \varphi|K \) gives us the graph of \( f|D \), which lies on a \( C^\infty \) arc.

### 6 Remarks on Arcs

Although the notion of strongly \( C^k \) is the one capturing the geometric notion of “smooth”, every polygonal path is weakly \( C^\infty \). Moreover, the standard formulas for
evaluating line integrals (e.g., \( \int_A \vec{F}(\vec{x}) \cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) \, dt \)) only require the path \( \vec{g}(t) \) to be weakly \( C^1 \); the arc \( A \) may have corners, with the velocity vector \( \vec{g}'(t) \) becoming zero at a corner.

Theorems 1.2, 1.3, and 1.6 produce strongly \( C^k \) arcs. In contrast, Theorem 1.5 produces a perfect set which meets all strongly \( C^2 \) arcs in a finite set. Theorem 1.7 shows that the weakly version of this theorem is false.

To prove Theorem 1.7, we begin with an interpolation result.

**Definition 6.1** An interpolation function is a \( \psi \in C([0, 1], [0, 1]) \) such that \( \psi(0) = 0 \) and \( \psi(1) = 1 \).

**Definition 6.2** Assume that \( D \) is a closed subset of \([0, 1]\) with \( 0, 1 \in D \). Fix \( g \in C(D, \mathbb{R}^n) \), and let \( \psi \) be an interpolation function. Then the \( \psi \) interpolation for \( g \) is the function \( \tilde{g} \in C([0, 1], \mathbb{R}^n) \) extending \( g \) such that whenever \((a, b)\) is a maximal interval in \([0, 1]\) \( \setminus D \) and \( u \in (a, b) \),

\[
\tilde{g}(u) = g(a) + (g(b) - g(a))\psi((u - a)/(b - a)).
\]

It is easily seen that \( \tilde{g} \) is indeed continuous on \([0, 1]\).

**Definition 6.3** Assume that \( D \) is a closed subset of \([0, 1]\) with \( 0, 1 \in D \). Then \( g \in C(D, \mathbb{R}^n) \) is flat iff for all \( \alpha \in \omega \), there is a bound \( M_\alpha \) such that for all \( u, t \in D \)

\[
\|g(u) - g(t)\| \leq M_\alpha|u - t|^\alpha.
\]

That is, \( g \) is flat iff for all \( \alpha \in \mathbb{N} = \omega \setminus \{0\} \), \( g \) is uniformly Lipschitz of order \( \alpha \) on \( D \). If \( D \) is finite, then every \( g : D \rightarrow \mathbb{R}^n \) is flat. If \( D \) contains an interval, then a flat \( g \) is constant on that interval, because it is Lipschitz of order 2 there; for \( t < t + h \) in the interval: \( \|g(t + h) - g(t)\| \leq k \cdot M_2 \cdot h^2/k^2 \) for all \( k \geq 1 \).

**Lemma 6.4** Assume that \( D \) is a closed subset of \([0, 1]\) with \( 0, 1 \in D \). Assume that \( g \in C(D, \mathbb{R}^n) \) is flat. Let \( \psi \) be an interpolation function such that \( \psi \in C^\infty([0, 1], [0, 1]) \) and \( \psi^{(k)}(0) = \psi^{(k)}(1) = 0 \) for all \( k \in \mathbb{N} \). Let \( \tilde{g} \) be the \( \psi \) interpolation for \( g \). Then \( \tilde{g} \in C^\infty([0, 1], \mathbb{R}^n) \) and \( \tilde{g}^{(k)}(t) = 0 \) for all \( t \in D \) and all \( k \in \mathbb{N} \).

**Proof.** It is sufficient to produce bounds \( B_k \) giving the following Lipschitz condition for all \( t \in D \) and \( u \notin D \):

1. \( \|\tilde{g}(u) - \tilde{g}(t)\| \leq B_0|u - t|^2. \)
2. \( \|\tilde{g}^{(k)}(u)\| \leq B_k|u - t|^2 \) for \( k \in \mathbb{N} \).
Note that (1)(2) fail for \( u, t \notin D \), since the derivatives there need not be 0. On the other hand, (1) holds for \( u, t \in D \), because \( g \) is flat.

Observe that (1) and 2-Lipschitz on \( D \) prove \( \tilde{g}'(t) = 0 \) for \( t \in D \), so that (2) makes \( \tilde{g} \in C^1([0,1], \mathbb{R}^n) \). For \( k \geq 2 \), induct on \( k \) to see that \( \tilde{g} \in C^{(k)}([0,1], \mathbb{R}^n) \): (2) for \( k - 1 \) and the fact that \( \tilde{g}^{(k-1)} \) is 2-Lipschitz on \( D \) prove \( \tilde{g}^{(k)}(t) = 0 \) for \( t \in D \), so (2) for \( k \) makes \( g^{(k)} \) continuous.

To prove (1)(2), assume, without loss of generality, \( t < u \). To handle (1)(2) together, let \( Q_0(u, t) = \|\tilde{g}(u) - \tilde{g}(t)\| \), and for \( k > 0 \), \( Q_k(u, t) = \|\tilde{g}^{(k)}(u)\| \). Consider the two cases:

Case I. \( (t, u) \cap D = \emptyset \): Say \( t = a < u < b \), where \( a, b \in D \) and \( (a, b) \) is a maximal interval in \([0,1]\setminus D\). So

\[
Q_k(u, t) = \|g(b) - g(a)\| \cdot |\psi^{(k)}(u) - \psi^{(k)}(t)| \cdot \frac{1}{(b-a)^k}.
\]

Let \( S_k \) be the largest value taken by the function \( |\psi^{(k)}| \). Consider:

Subcase I.1. \( (b-a)^2 \leq (u-a) \): Here,

\[
Q_k(u, t) \leq \|g(b) - g(a)\| \cdot S_k \cdot \frac{1}{(b-a)^k} \cdot \frac{(u-a)^2}{(u-a)^2} \leq M_k + 4 S_k (u-a)^2.
\]

Subcase I.2. \( (b-a)^2 \geq (u-a) \): In this case, use Taylor's Theorem and the assumption \( \psi^{(n)}(0) = 0 \), for all \( n \in \mathbb{N} \), to bound \( |\psi^{(k)}(z)| \) by \( S_{2k+4} \frac{1}{(k+4)!} z^4 \). Then,

\[
Q_k(u, t) \leq M_0 \cdot \left| \psi^{(k)} \left( \frac{u-a}{b-a} \right) \right| \cdot \frac{(b-a)^{k+4}}{(u-a)^{k+4}} \cdot \frac{(u-a)^{k+4}}{(b-a)^{2k+4}} \leq M_0 \cdot \frac{S_{2k+4}}{(k+4)!} (u-a)^2.
\]

Case II. \( (t, u) \cap D \neq \emptyset \): Let \( a = \sup(D \cap [t, u]) \), so \( t < a < u \) and Case I applies to \( a, u \). For (1), use the fact that \( g \) is flat, together with

\[
\|\tilde{g}(u) - \tilde{g}(t)\| \leq \|\tilde{g}(u) - \tilde{g}(a)\| + \|g(a) - g(t)\|.
\]

For (2), \( \|\tilde{g}^{(k)}(u)\| \leq B_k |u-a| \leq B_k |u-t|^2 \). \( \square \)

**Proof of Theorem 1.7.** Passing to a subset, and possibly translating it, let \( E = \{ \bar{x}_j : j \in \omega \} \), where the \( \bar{x}_j \) converge to \( \bar{0} \), and

a. \( \|\bar{x}_0\| > \|\bar{x}_1\| > \|\bar{x}_2\| > \cdots \).

b. \( \|\bar{x}_j\| \leq 2^{-j^2} \) for each \( j \).

Let \( A \) be the set obtained by connecting each \( \bar{x}_j \) to \( \bar{x}_{j+1} \) by a straight line segment; so \( A \) is a “polygonal” arc, with \( \omega \) steps. Moreover, the natural path which traverses it from \( \bar{0} \) to \( \bar{x}_0 \) will be 1-1, because (a) guarantees that the line segments forming \( A \) meet only at the \( \bar{x}_j \). Let \( D = \{0\} \cup \{2^{-j} : j \in \omega \} \), and define \( g : D \to \mathbb{R}^n \) by \( g(0) = \bar{0} \) and \( g(2^{-j}) = \bar{x}_j \). Then \( g \) is flat, by (b) (with \( M_\alpha = 2^{1+\alpha+\alpha^2} \)).

Let \( \psi \in C^\infty(\mathbb{R}) \) be such that
\( \psi(t) = 0 \) when \( t \leq 0 \) and \( \psi(t) = 1 \) when \( t \geq 1 \).

\( \psi'(t) > 0 \) for \( 0 < t < 1 \).

\( \psi^{(k)}(0) = \psi^{(k)}(1) = 0 \) for \( k \geq 1 \).

Such a \( \psi \) may be obtained by integrating a scalar multiple of the function described in Lemma 5.1. Let \( \tilde{g} : [0, 1] \to \mathbb{R}^n \) be the \( \psi \) interpolation for \( g \). Then, by Lemma 6.4, \( \tilde{g} \in C^\infty([0, 1], \mathbb{R}^n) \).

For the path \( \tilde{g} \) in the preceding proof, all \( \tilde{g}^{(k)} \) (for \( k \geq 1 \)) will be \( \vec{0} \) when passing through each \( \vec{x}_j \), so that no acceleration is felt when rounding a corner. Also, each \( \tilde{g}^{(k)} \) will be \( \vec{0} \) at \( t = 0 \).

Now consider the perfect set version.

**Theorem 6.5** If \( E \subseteq \mathbb{R}^n \) is Borel and uncountable, then \( E \) meets some weakly \( C^\infty \) arc in an uncountable set.

**Proof.** Write elements of \( \mathbb{R}^n \) as \( \vec{x} = (x^1, \ldots, x^n) \). By shrinking and rotating \( E \), we may assume that \( E \) is a Cantor set and the projection \( \pi^1 \) of \( E \) on the \( x^1 \) coordinate is 1-1. Shrinking \( E \) further, we may assume that \( E = \bigcap_j (\bigcup \{ F_{\sigma} : \sigma \in \{0, 2\}^j \}) \), where the \( F_{\sigma} \) are compact and form a tree and each \( \text{diam}(F_{\sigma}) \leq 3^{-(\text{lh}(\sigma))^2} \).

In \( \mathbb{R} \), the "\( t \)-axis", let \( D \) be the usual middle-third Cantor set. Then \( D = \bigcap_j (\bigcup \{ I_{\sigma} : \sigma \in \{0, 2\}^j \}) \), where \( I_{\sigma} \) is an interval of length \( 3^{-(\text{lh}(\sigma))} \). Let \( g : D \to E \) be the natural homeomorphism. So, if \( \alpha \in \{0, 2\}^\omega \), it determines the point \( t_\alpha = \sum_{i \in \omega} (\alpha_i 3^{-i}) \in D \). Then \( \bigcap_{i \in \omega} I_{\alpha_i} = \{t_\alpha\} \) and \( \bigcap_{i \in \omega} F_{\alpha_i} = \{g(t_\alpha)\} \).

Note that \( g \) is flat. Let \( \psi \in C^\infty(\mathbb{R}) \) be as in the proof of Theorem 1.7, and let \( \tilde{g} \) be the \( \psi \) interpolation for \( g \). Then \( \tilde{g} \in C^\infty([0, 1], \mathbb{R}^n) \).

Finally, in choosing \( E \) and the \( F_{\sigma} \), make sure that if \( \sigma < \tau \) lexicographically, then all elements of \( \pi^1(F_{\sigma}) \) are less than all elements of \( \pi^1(F_{\tau}) \). This will guarantee that \( \pi^1 \circ g : D \to \mathbb{R} \) is order-preserving, so that \( \tilde{g} \) is a 1-1 function.

Under MA(\( \aleph_1 \)), if \( E \subseteq \mathbb{R}^n \) has size \( \aleph_1 \), then \( E \) can be covered by \( \aleph_0 \) weakly \( C^\infty \) arcs. In particular, \( E \) can be covered by \( \aleph_0 \) copies, or rotated copies, of the perfect set \( g(D) \) constructed in the preceding proof.

**References**


