Weak Measure Extension Axioms *

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Abstract

We consider axioms asserting that Lebesgue measure on the real line may be extended to measure a few new non-measurable sets. Strong versions of such axioms, such as real-valued measurability, involve large cardinals, but weak versions do not. We discuss weak versions which are sufficient to prove various combinatorial results, such as the non-existence of Ramsey ultrafilters, the existence of ccc spaces whose product is not ccc, and the existence of S- and L- spaces. We also prove an absoluteness theorem stating that assuming our axiom, every sentence of an appropriate logical form which is forced to be true in the random real extension of the universe is in fact already true.

1 Introduction.

In this paper, a *measure* μ on a set X is a countably additive measure whose domain (the μ-measurable sets) is some σ-algebra of subsets of X.

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We are primarily interested in finite measures, although most of our results extend to σ-finite measures in the obvious way. By the Axiom of Choice (which we always assume), there are subsets of $[0,1]$ which are not Lebesgue measurable. In an attempt to measure them, it is reasonable to postulate measure extension axioms of the following form:

Definition 1.1 *If θ is any cardinal and μ is a measure on the set X, then $ME_\theta(X,\mu)$ holds iff whenever we choose a family $\mathcal{E}$ of θ or fewer subsets of X, there is a measure ν on X which extends μ such that each set in $\mathcal{E}$ is ν-measurable. ME_θ denotes $ME_\theta([0,1],\lambda)$, where λ is Lebesgue measure on $[0,1]$.*

For $\theta < \omega$, $ME_\theta(X,\mu)$ holds for every finite measure μ, but for infinite θ, it can depend on (X,μ) and the underlying model of set theory. Regardless of the set theory, there is always some separable atomless probability space (X,μ) such that $ME_\omega(X,\mu)$ is false (by Theorem 4 of Grzegorek [10]; see also §6). In this paper, we are concerned mainly with ME_θ, not arbitrary $ME_\theta(X,\mu)$, but in applications of ME_θ, it is often convenient to replace $[0,1]$ by 2^ω or by $[0,1]^\omega$ (with the usual product measure). This is justified by the following:

Proposition 1.1 *Let μ be a finite Borel measure on the compact metric space X. Then ME_θ implies $ME_\theta(X,\mu)$. Furthermore, ME_θ is equivalent to $ME_\theta(X,\mu)$ unless μ is a countable sum of point masses.*

Proof: To derive $ME_\theta(X,\mu)$ from ME_θ, let $f:[0,1]\to X$ be a Borel measurable function such that μ is the induced measure, $c\cdot\lambda f^{-1}$, where $c=\mu(X)$. Then we can extend μ to measure a family $\mathcal{E}$ of subsets of X by extending λ to measure $\{f^{-1}(E): E\in\mathcal{E}\}$. Conversely, if μ is not a sum of point masses, we can fix a closed $K\subseteq X$ of positive measure such that μ restricted to K is atomless. We can then derive ME_θ from $ME_\theta(X,\mu)$ using a function $g:K\to[0,1]$ such that $\lambda = c\cdot\mu g^{-1}$, where $c=1/\mu(K)$. ∎

Note that if μ is a countable sum of point masses, then $ME_\theta(X,\mu)$ is a triviality, since then every subset of X is μ-measurable.

Now, consider ME_θ for various infinite θ. ME_ω is false under CH or MA (for numerous reasons – see below). Nevertheless, ME_ω and also ME_{ω_1} are consistent with $\mathfrak{c}$ (the continuum, 2^ω) being ω_2. In general, for uncountable θ, one can get ME_θ together with $\mathfrak{c} = \theta^+$:

Theorem 1.2 (Carlson [3]) *Assume that in the ground model V, θ is some infinite cardinal with $\theta^\omega = \theta$. Let $V[G]$ be formed from V by adding θ^+ or more random reals. Then ME_θ holds in $V[G]$.*

In particular, if CH holds in V and $\theta = \omega_1$, then we get a model of ME_{ω_1} plus $\mathfrak{c} = \omega_2$. More generally, for regular $\theta > \omega$, we can get models of ME_θ with either $\mathfrak{c} = \theta^+$ or $\mathfrak{c} > \theta^+$, Furthermore, for small θ (e.g., below the first weakly Mahlo cardinal), ME_θ implies MA_θ for the partial order which adds one random real (see Corollary 2.9), and hence $\mathfrak{c} \geq \theta^+$. Carlson's paper [3] discusses applications of $ME_\theta(2^\kappa, \mu)$ for various uncountable cardinals κ (where μ is the usual product measure) to normal Moore space problems, whereas our paper concentrates on applications of ME_θ (that is, $ME_\theta(2^\omega, \mu)$).

The emphasis of this paper is on small θ, but we remark briefly on ME_θ for larger values, which leads naturally to large cardinal axioms. By the method of Solovay [22] (see also [3]), the assumption of ME_θ plus $\mathfrak{c} \leq \theta$ is equiconsistent with a weakly compact cardinal . By Ulam [26], the existence of a real-valued measurable cardinal is equivalent to what one might call ME_∞; that is, Lebesgue measure can be extended to measure all sets of reals simultaneously. So, by Solovay [22], ME_∞ is equiconsistent with the existence of a (two-valued) measurable cardinal. For a discussion of $PMEA$, which involves extending measures on various 2^κ, see Fleissner [7].

We turn now to applications of ME_θ. These are all statements which hold in random real extensions, and would thus would be easy to prove from a real-valued measurable cardinal, using Solovay's Boolean ultrapower method [22], but require some care to derive from the weaker ME_θ. In §3, we establish an absoluteness theorem which says that, assuming ME_θ, if a statement φ of a certain simple logical form is true about θ in random real extensions, then φ is already true in V. The form of φ enables us to produce in V objects which can be constructed from a single random real. Some applications are given by the following theorem.

Theorem 1.3 *ME_{ω_1} implies*

1. *there are ccc topological spaces X and Y such that $X \times Y$ is not ccc;*
2. *there are strong S- and L-spaces;*
3. *there is an uncountable $\aleph_1$-entangled set.*

As usual, *ccc* denotes the countable chain condition. By Galvin [9], CH implies that *ccc* is not productive, whereas $MA + \neg CH$ implies that *ccc* is productive (see, e.g., Theorem 2.24 of [15]). As is well known (see, e.g., Exercise 8.C1 of [15]), productivity of *ccc* is the same whether we deal with topological spaces or with partial orders, and work on productivity of *ccc* usually deals with the partial orders directly.

Roitman showed [20] that in random real extensions of V, *ccc* partial orders $\mathbb{P}$ and $\mathbb{Q}$, with $\mathbb{P} \times \mathbb{Q}$ not *ccc*, may be constructed from a single random real. In the same paper, she constructed strong S- and L-spaces from a single random real. The fact that uncountable entangled sets are added by a random real is due to Todorčević (see page 55 of [25] for a proof). It follows almost immediately from these facts and Solovay's Boolean ultrapower construction [22] that the conclusions to Theorem 1.3 follow from a real-valued measurable cardinal; see also Fremlin [8] for a discussion. To apply our absoluteness result to produce these results from the weaker assumption of ME_{ω_1}, however, we exploit the form of the construction of the desired objects from the random real; this is discussed in §3. Actually, Theorem 1.3(1) follows directly from Theorem 1.3(3), by Todorčević [24].

While our absoluteness result applies to objects which can be constructed from one random real, some further applications of ME_θ do not seem to fit this pattern. For example, ME_ω implies MA_{ω_1} for the partial order which adds one random real (Corollary 2.5), which in turn has a number of well-known consequences (e.g., every subset of $[0, 1]$ of size ω_1 is of first category). Of course, this refutes CH. Also (Corollary 6.2), ME_ω implies that no Lebesgue measurable set of positive measure can be an increasing union of Lebesgue nullsets. This also refutes CH, as well as full MA. Finally, we mention:

Theorem 1.4 *ME_ω implies that there are no Ramsey ultrafilters.*

A *Ramsey* (or, selective) *ultrafilter* is a nonprincipal ultrafilter $\mathcal{U}$ on ω such that every partition $\Phi : [\omega]^2 \to 2$ has a homogeneous set in $\mathcal{U}$. As is well known [2], this implies that for each finite n, $\mathcal{U}$ is also Ramsey for partitions on n-tuples. Under CH or MA, there is a Ramsey ultrafilter, as is easily seen by W. Rudin's construction [21] of a P-point ultrafilter. It is already known [14] that there are no Ramsey ultrafilters in the model obtained by adding at least $\mathfrak{c}^+$ random reals. Using this method of proof, Fremlin [8] shows that a real-valued measurable cardinal refutes the existence of Ramsey ultrafilters. A proof of Theorem 1.4 may be patterned after the argument in [8], but we give a different argument, which also improves the result of [14] to:

Theorem 1.5 *Let $V[G]$ be formed from V by adding ω_2 or more random reals. Then in $V[G]$, there are no Ramsey ultrafilters.*

Although the method of [14],[8] alone does not seem to prove this, in proving Theorems 1.4 and 1.5, we emulate [14],[8] to refute a property weaker than Ramsey, known as "rapid P-point" or "semi-selective".

A special case of Theorem 1.2 is that ME_ω becomes true if we add $\mathfrak{c}^+$ random reals. But while adding ω_2 random reals suffices for Theorem 1.5, adding ω_2 random reals may not be enough to get ME_ω. To see this, observe that ME_ω is false if the well-order on the cardinal $\mathfrak{c}$ is in the σ-algebra generated by rectangles, since, by Fubini's Theorem, the sides of the rectangles will form a countable collection of subsets of $\mathfrak{c}$ (equivalently, of $[0, 1]$) which cannot be measured by any atomless σ-additive probability measure. In particular, by [12] or [19], ME_ω is false under CH or MA. Furthermore, suppose the ground model V satisfies $MA + \neg CH$. Then, adding ω_2 random reals does not change $\mathfrak{c}$, so it is still true in $V[G]$ that the well-order on $\mathfrak{c}$ is in the σ-algebra generated by rectangles, so ME_ω is false in $V[G]$. Further use of rectangles to derive theorems from ME_ω occurs in §2.

We also cannot replace the ω_2 by ω_1 in Theorem 1.5: if the ground model satisfies CH, then CH will remain true after adding ω_1 random reals, so there will be a Ramsey ultrafilter in the extension.

Theorems 1.4 and 1.5 both have the same conclusion, "no Ramsey ultrafilters". The proofs, given in §4, are similar too, and utilize the same probabilistic argument, although the proof of Theorem 1.5 adds a forcing ingredient. Our method for refuting Ramsey ultrafilters in §4 may seem a bit artificial, since the argument does not deal directly with the Ramsey property at all, but rather with a rather technical consequence thereof. In §5, we present a more natural argument using random graph theory. Actually, the method in §5 requires more work in verifying the details than does the method of §4, but it derives the lack of Ramsey ultrafilters directly from a lemma about random graphs on finite sets, which might be of some interest in its own right.

2 Preserving Suprema.

A key ingredient of §3's proof of our absoluteness result is the existence of a measure algebra in which certain suprema are preserved. We begin this section by reviewing some basic facts about measure algebras, and then we look at the suprema preserving strength of ME_θ for various θ.

Definition 2.1 *If ν is a probability measure on a set X, then $\mathbb{B}(\nu)$ is the measure algebra of the ν-measurable subsets of X modulo the ν-nullsets. If κ is a cardinal, and ν_κ is the usual product measure on 2^κ, then we abbreviate $\mathbb{B}(\nu_\kappa)$ by $\mathbb{B}_\kappa$. If $\mathcal{E}$ is a family of ν-measurable sets, $\bigvee \mathcal{E}$ abbreviates $\bigvee_{E\in\mathcal{E}}[E]$.*

Note that the elements of $\mathbb{B}(\nu)$ are equivalence classes $[E]$ of ν-measurable sets, with $[D] \leq [E]$ iff $D\backslash E$ is a nullset. $\mathbb{B}_\kappa$ is the measure algebra with which one forces to add κ random reals. This is equivalent to forcing with the Baire sets of positive measure, as in [17]. When just doing forcing, it is somewhat simpler to use the Baire sets, rather than their equivalence classes, but when discussing algebraic properties, such as suprema, it is somewhat simpler to work with the Boolean algebra.

As is well known, $\mathbb{B}(\nu)$ is a complete Boolean algebra, and the following lemma relates suprema with unions:

Lemma 2.1 *If $\mathcal{E}$ is a family of ν-measurable sets, then $\bigvee \mathcal{E} = [\bigcup \mathcal{E}_0]$ for some countable $\mathcal{E}_0 \subseteq \mathcal{E}$.*

If $\mathcal{E}$ is uncountable, then $\bigcup \mathcal{E}$ may fail to be measurable. If it is measurable, then $\bigvee \mathcal{E} \leq [\bigcup \mathcal{E}]$, but this inequality may be strict; for example, let $\mathcal{E}$ be a family of singletons from $[0, 1]$.

Definition 2.2 *If $\mathcal{E}$ is a family of ν-measurable sets, with $|\mathcal{E}| = \theta$, then ν* preserves the θ-supremum *$\bigvee \mathcal{E}$ iff $\bigcup \mathcal{E}$ is ν-measurable and $\bigvee \mathcal{E} = [\bigcup \mathcal{E}]$ in $\mathbb{B}(\nu)$.*

First we look at preserving ω_1-suprema, and then we look at a not-so-large cardinal property to handle preserving θ-suprema.

Theorem 2.2 *If $\mathcal{E} \subseteq \mathcal{P}(X)$ and $|\mathcal{E}| = \omega_1$, then there is a countable collection $\mathcal{S} \subseteq \mathcal{P}(X)$ such that every measure on X which measures $\mathcal{E} \cup \mathcal{S}$ preserves $\bigvee \mathcal{E}$.*

We shall prove this after pointing out two corollaries. Note first that this theorem applies when extending a measure μ with $\mathcal{E} \subseteq dom(\mu)$:

Corollary 2.3 *$ME_\omega(X, \mu) \Rightarrow X$ is not a union of ω_1 μ-nullsets.*

For Lebesgue measure, this gives us a form of Martin's Axiom. Specifically, let $MA_\theta(1rr)$ represent MA_θ for random real forcing; that is, $MA_\theta(1rr)$ says that whenever $\mathbb{P}$ is a partial order for adding one random real, and $\mathcal{D}$ is a family of no more than θ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ meeting each $D \in \mathcal{D}$. The following well-known proposition relates the preservation of θ-suprema to $MA_\theta(1rr)$.

Proposition 2.4 *$MA_\theta(1rr)$ is equivalent to the statement that $[0, 1]$ is not the union of θ Lebesgue nullsets.*

Thus, by Corollary 2.3, we have the following.

Corollary 2.5 *$ME_\omega \Rightarrow MA_{\omega_1}(1rr)$.*

In particular, $ME_\omega \Rightarrow \mathfrak{c} \geq \omega_2$. Note that we cannot "raise the cardinals by 1" in Corollary 2.5: ME_{ω_1} is consistent with $\mathfrak{c} = \omega_2$, so it cannot imply $MA_{\omega_2}(1rr)$. Hence, in Theorem 2.2, if $|\mathcal{E}| = \omega_2$, we cannot expect to get $|\mathcal{S}| = \omega_1$. We can get an $\mathcal{S}$ of size ω_2, as we explain after proving Theorem 2.2.

Our proof employs the following ancient fact (proved in Kunen [12] and Rao [19]) about $\omega_1 \times \omega_1$.

Theorem 2.6 *Each subset of $\omega_1 \times \omega_1$ is in a σ-algebra generated by countably many rectangles of the form $B \times C$, where $B, C \subseteq \omega_1$.*

Another easy fact we employ in proving Theorem 2.2 reduces suprema to disjoint suprema:

Lemma 2.7 *Suppose that E_α is ν-measurable for each $\alpha < \omega_1$. Let $E'_\alpha = E_\alpha \setminus \bigcup_{\beta<\alpha} E_\beta$ (so $E'_0 = E_0$). Then $\bigvee_{\alpha<\omega_1}[E_\alpha] = \bigvee_{\alpha<\omega_1}[E'_\alpha]$.*

Now we preserve our ω_1-suprema:

Proof of Theorem 2.2: Fix a collection $\mathcal{E} = \{E_\alpha : \alpha < \omega_1\}$. By Lemma 2.7, we may assume that $\mathcal{E}$ is a disjoint collection. By Theorem 2.6, the set $L = \{\langle \beta, \alpha \rangle : \alpha < \beta < \omega_1\}$ below the diagonal is in the σ-algebra generated by a countable collection $\{B_n \times C_n : n < \omega\}$, where each B_n and C_n is a subset of ω_1. Let $E = \bigcup_{\alpha<\omega_1} E_\alpha$. Define $F : E \to \omega_1$ so that $F(x) = \alpha$ iff $x \in E_\alpha$. Let

$$\mathcal{S} = \{E\} \cup \{F^{-1}(B_n) : n \in \omega\} \cup \{F^{-1}(C_n) : n \in \omega\} \ .$$

Let $L' = \bigcup_{\alpha<\beta<\omega_1} E_\beta \times E_\alpha$. Observe that L' is in the σ-algebra generated by the $F^{-1}(B_n) \times F^{-1}(C_n)$.

By way of contradiction, assume that ν measures $\mathcal{E} \cup \mathcal{S}$ but fails to preserve $\bigvee \mathcal{E}$; that is, $\bigvee_{\alpha<\omega_1}[E_\alpha] < [E]$. By Lemma 2.1, fix $\gamma < \omega_1$ such that $\bigvee_{\alpha<\omega_1}[E_\alpha] = \bigvee_{\alpha<\gamma}[E_\alpha]$. Let $G = \bigcup\{E_\alpha : \gamma \le \alpha < \omega_1\}$. Then G is measurable and not null, but is partitioned into the E_α (for $\gamma \le \alpha < \omega_1$), which are null. Define $M = \bigcup\{E_\beta \times E_\alpha : \gamma \le \alpha < \beta < \omega_1\}$. Note that L', and hence also M, is measurable in the product measure $\nu \times \nu$. Each vertical slice $M_x = \{y : (x,y) \in M\}$ is contained in a countable union of the nullsets E_α, and is hence a nullset, whereas each horizontal slice $M^y = \{x : (x,y) \in M\}$ contains all but countably many of the E_α (for $\gamma \le \alpha < \omega_1$), so $\nu(M^y) = \nu(G) > 0$. But then M contradicts Fubini's Theorem. ∎

We turn now to preserving θ-suprema for $\theta > \omega_1$. In order to pin down the θ for which ME_θ guarantees that we can preserve these suprema, we define a not-so-large cardinal property.

In the following, $\mathcal{L}$ is always a countable first order language containing the symbol "$<$". For any ordinal δ, a structure $\mathfrak{A}$ for $\mathcal{L}$ is called a *δ-structure* iff $\mathfrak{A}$ has universe δ and "$<$" is interpreted as the usual well order on δ. Another structure $\mathfrak{B}$ for $\mathcal{L}$ is called an *end extension* of $\mathfrak{A}$ iff $\mathfrak{B}$ is a proper extension of $\mathfrak{A}$ and no member of δ gets a new element, that is, if $\alpha < \delta$ and $b \in B$, then $b <_{\mathfrak{B}} \alpha$ implies $b \in \delta$; if in addition, $\mathfrak{B}$ is an elementary extension, it is called an *elementary end extension*, or *eee*. See [4] for general background on elementary end extensions. We consider here *real-valued elementary end extendible* ordinals:

Definition 2.3 *An ordinal δ has the* rveee property*, or is* rveee*, iff each δ-structure $\mathfrak{A}$ has an eee in some random real extension of the universe.*

The Compactness Theorem implies that ω is rveee, and the end extension is obtainable without adding random reals. We shall see presently (Proposition 2.12) that a rveee ordinal is, in fact, a cardinal; it is also weakly Mahlo, and is weakly compact in L. Hence, the following theorem applies to preserving θ-suprema for a "reasonable" number of smaller θ:

Theorem 2.8 *If $\mathcal{E} \subseteq \mathcal{P}(X)$ and each uncountable $\delta \leq |\mathcal{E}|$ is not rveee, then there is a family $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $|\mathcal{S}| = |\mathcal{E}|$, and such that every measure on X which measures $\mathcal{E} \cup \mathcal{S}$ preserves $\bigvee \mathcal{E}$.*

Again, by Proposition 2.4, we have the following.

Corollary 2.9 *For $\theta \geq \omega_1$ such that each uncountable $\delta \leq \theta$ is not rveee, $ME_\theta \Rightarrow MA_\theta(1rr)$.*

Since $MA_{\mathfrak{c}}(1rr)$ is false and there are models of ME_θ plus $\mathfrak{c} = \theta^+$ (see Theorem 1.2), we cannot preserve θ^+-suprema using just ME_θ. Similarly, if $\theta = \mathfrak{c}$ is real-valued measurable, then ME_θ is true and $MA_\theta(1rr)$ is false, so the assumption on rveee ordinals cannot be dropped from the corollary or the theorem.

We prove Theorem 2.8 by contradiction; assuming $\bigvee \mathcal{E} < [\bigcup \mathcal{E}]$ enables us to use an ultrapower in some random real extension to build an eee. The following lemma simplifies the construction:

Lemma 2.10 *Suppose there are no uncountable rveee ordinals $\delta \leq \theta$. Then there is a θ-structure $\mathfrak{A}$ such that every proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ contains a nonstandard integer.*

Proof: Let $\mathfrak{A}$ encode, for each uncountable ordinal $\delta \leq \theta$, a δ-structure $\mathfrak{A}_\delta$ witnessing that δ is not rveee. ∎

To prove Theorem 2.8, we shall use an ultrapower based on definable sets. For a θ-structure $\mathfrak{A}$, let $\mathcal{D}_\mathfrak{A}$ be the set of subsets of θ first order definable in $\mathfrak{A}$ from elements of θ, and let $\mathcal{F}_\mathfrak{A}$ be the set of functions in θ^θ first order definable in $\mathfrak{A}$ from elements of θ. For any ultrafilter $\mathcal{U}$ on $\mathcal{D}_\mathfrak{A}$, we have the ultrapower $\mathcal{F}_\mathfrak{A}/\mathcal{U}$. A version of Łoś' Theorem for definable functions shows that $\mathcal{F}_\mathfrak{A}/\mathcal{U}$ is an elementary extension of $\mathfrak{A}$. Although the original Łoś' Theorem does not apply directly here, one can prove the version we need simply by mirroring the original's proof, which inducts on the complexity of formulas, taking care in the existential case to produce a *definable* function. The extension $\mathcal{F}_\mathfrak{A}/\mathcal{U}$ will be proper iff $\mathcal{U}$ is nonprincipal. Also, $\mathcal{F}_\mathfrak{A}/\mathcal{U}$ will be an ω-model iff for each function $f \in \mathcal{F}_\mathfrak{A}$ with $f : \theta \to \omega$, there is some $n \in \omega$ so that $f^{-1}(\{n\}) \in \mathcal{U}$.

Proof of Theorem 2.8: Fix $\mathcal{E} \subseteq \mathcal{P}(X)$ such that each uncountable $\delta \leq |\mathcal{E}|$ is not rveee, and let $\theta = |\mathcal{E}|$. Let $\{E_\alpha : \alpha < \theta\}$ list $\mathcal{E}$, and form the corresponding disjoint family in the usual way: for each $\alpha < \theta$, set $E'_\alpha = E_\alpha \setminus \bigcup_{\beta<\alpha} E_\beta$. Let $E = \bigcup_{\alpha<\theta} E_\alpha = \bigcup_{\alpha<\theta} E'_\alpha$, and define $\phi : E \to \theta$ so that $\phi(x) = \alpha$ for each $x \in E'_\alpha$. Fix $\mathfrak{A}$ as in Lemma 2.10, and let

$$\mathcal{S} = \{E\} \cup \{E'_\alpha : \alpha < \theta\} \cup \{\phi^{-1}(D) : D \in \mathcal{D}_\mathfrak{A}\} \quad .$$

Suppose ν measures $\mathcal{S} \cup \mathcal{E}$, and suppose that ν does not preserve $\bigvee \mathcal{E}$, so $[E] > \bigvee_{\alpha<\theta}[E_\alpha] \geq \bigvee_{\alpha<\theta}[E'_\alpha]$ in $\mathbb{B}(\nu)$. We shall derive a contradiction.

Let G be a $\mathbb{B}(\nu)$-generic filter over V with $[E] \in G$ but $\bigvee_{\alpha<\theta}[E'_\alpha] \notin G$. Define $\mathcal{U} = \{D \subseteq \theta : [\phi^{-1}(D)] \in G\}$. By upward closure of G, each $[E'_\alpha] \notin G$, so $\mathcal{U}$ is nonprincipal, and hence $\mathcal{F}_\mathfrak{A}/\mathcal{U}$ is a proper extension of $\mathfrak{A}$.

Finally, we show that $\mathcal{F}_\mathfrak{A}/\mathcal{U}$ is an ω-model, which will contradict the choice of $\mathfrak{A}$. Suppose $f \in \mathcal{F}_\mathfrak{A}$ and $f : \theta \to \omega$. Then θ is the disjoint union of the sets $C_n = \{\alpha < \theta : f(\alpha) = n\}$. Since the $[\phi^{-1}(C_n)]$ form a partition of unity in the ground model V, some $[\phi^{-1}(C_n)]$ is in G, so some C_n is in $\mathcal{U}$. ∎

Next, we show that rveee ordinals are weakly Mahlo in V, and weakly compact in L. This argument uses nothing special about random real forcing besides the fact that cofinalities are preserved.

First, observe that the Keisler-Silver [11] argument works also for forcing extensions:

Lemma 2.11 *Let δ be any uncountable regular cardinal, and let $\mathbb{P}$ be any forcing order such that δ remains regular in $\mathbb{P}$-generic extensions. If each δ-structure has an eee in some $\mathbb{P}$-extension of the universe, then each δ-structure has a well-ordered eee in some $\mathbb{P}$-extension of the universe.*

Proposition 2.12 *If $\delta > \omega$ and each δ-structure has an eee in some cofinality-preserving forcing extension, then δ is weakly Mahlo and is weakly compact in L.*

Proof: As is well known, if δ fails to be weakly Mahlo or fails to be weakly compact in L, then there is a δ-structure $\mathfrak{A}$ which has no eee in V. We simply observe that this $\mathfrak{A}$ continues to work in cofinality-preserving forcing extensions. Specifically, we consider cases:

If δ fails to be a regular cardinal, let $cf(\delta) = \beta < \delta$, and set $\mathfrak{A} = \langle \delta; <, g \rangle$, where $g \restriction \beta$ is a cofinal map from β to δ, and $g(\xi) = 0$ for $\beta \leq \xi < \delta$. Then $\mathfrak{A}$ will fail to have an eee in *every* extension of V.

If δ is regular but not weakly Mahlo, let $C \subset \delta$ be a club which contains only singular limit ordinals. For each $\xi < \delta$, choose a cofinal map $h_\xi : cf(\xi) \to \xi$. Code these by a map $H : \delta \times \delta \to \delta$ by letting $H(\xi, \zeta) = h_\xi(\zeta)$ when $\zeta < cf(\xi) < \xi \in C$, and $H(\xi, \zeta) = 0$ otherwise. Let $\mathfrak{A} = \langle \delta; <, C, H \rangle$. Then $\mathfrak{A}$ will fail to have a well-founded eee in every extension of V in which δ is regular.

If δ is regular but not weakly compact in L, then in L, there is a δ-structure $\mathfrak{A}$ such that $\mathfrak{A}$ has no *eee* in any extension of L in which δ remains regular. ∎

By a similar proof, one may show that if S is stationary in δ, then $S \cap \alpha$ is stationary in α for some *regular* $\alpha < \delta$. This property was also discussed in [16]; it directly implies the usual weak-inaccessible type large cardinal properties one gets from a weakly compact cardinal.

It is easy to see that every weakly compact cardinal is rveee in every random real extension, so that a rveee cardinal $\leq \mathfrak{c}$ is equiconsistent with a weakly compact cardinal. In fact, if δ is weakly compact in V, and $V[G]$ is formed by adding *more than* δ random reals *or* Cohen reals, then δ is actually *2veee* in $V[G]$; that is every δ-structure has an eee in some the universe (now, $V[G]$) itself (see [13]). In particular, considering the Cohen real case, we see that the existence of an rveee cardinal cannot have any interesting measure-theoretic consequences.

Corollary 2.13 *$ME_{\mathfrak{c}}$ implies that some $\delta \leq \mathfrak{c}$ is weakly compact in L.*

Proof: If not, then by Proposition 2.12, each uncountable $\delta \leq \mathfrak{c}$ fails to be rveee, so by Corollary 2.9, $MA_{\mathfrak{c}}(1rr)$ holds, which is impossible. ■

So, as Carlson notes in [3] by a somewhat different argument, $ME_{\mathfrak{c}}$ is equiconsistent with the existence of a weakly compact cardinal.

3 An Absoluteness Result.

In this section, we describe our absoluteness result for random real models and use it to prove Theorem 1.3. As in §2, we view random real models as extensions by some measure algebra $\mathbb{B}_\kappa$. The following additional definitions will be useful. If φ is a sentence in the forcing language, $\|\varphi\|$ denotes its Boolean value (the maximum condition which forces φ). Each $b \in \mathbb{B}_\kappa$ has a countable support, $supt(b)$, which is the minimal subset $T \subseteq \kappa$ such that b is the equivalence class of a cylinder over T. If τ is a $\mathbb{B}_\kappa$-name (in the forcing language), $supt(\tau)$ denotes the union of all $supt(b)$ such that b is used (hereditarily) in the construction of τ. Whenever we name reals or subsets of ω, we choose names with countable support.

Theorem 3.1 *Fix $\theta \geq \omega_1$ such that each uncountable $\alpha \leq \theta$ is not rveee, and assume ME_θ. Suppose*

$$1 \Vdash_{\mathbb{B}_\theta} (\exists x : \omega \to 2)\ (\forall Y \subseteq \theta)\ \varphi(x, Y)\ , \tag{1}$$

where φ is a first order formula over a θ-structure $\mathfrak{A}$. Then it is true (in V) that

$$(\exists x : \omega \to 2)\ (\forall Y \subseteq \theta)\ \varphi(x, Y)\ \ . \tag{2}$$

Some remarks on syntax: The notion of θ-structure is as in §2, but x and Y are second order variables here, so that equation (2) expresses a special kind of Σ^1_2 property of the cardinal θ plus whatever function and relation constants are contained in $\mathfrak{A}$. Since $\mathfrak{A}$ is fixed here, we write "φ" instead of "$\mathfrak{A} \models \varphi$". If θ is real-valued measurable, then by the method of [22], Theorem 3.1 holds for all Σ^1_n properties.

Of course, the theorem holds also for $\theta = \omega$, but is trivial in that case by Shoenfield's Theorem. In proving the theorem, the following definitions will be useful.

Definition 3.1 *If* $Y \subseteq 2^\omega \times \theta$, *then* $Y^\alpha = \{x \in 2^\omega : (x, \alpha) \in Y\} \subseteq 2^\omega$ *(for each* $\alpha \in \theta$*), and* $Y_x = \{\alpha \in \theta : (x, \alpha) \in Y\} \subseteq \theta$ *(for each* $x \in 2^\omega$*).*

Definition 3.2 $\dot{r}$ *is the name for the* official random real. *That is,* $1 \Vdash \dot{r} : \omega \to 2$, *and* $\|\dot{r}(n) = \ell\| = [\{f : f(n) = \ell\}]$ *(for* $n < \omega$, $\ell < 2$*).*

The notation $\dot{r}$ is used both when we are forcing with some $\mathbb{B}_\kappa$, for $\kappa \geq \omega$, and when we are forcing with a $\mathbb{B}(\nu)$, where ν is some measure on 2^ω extending Lebesgue measure.

The following lemma lets us drop a quantifier in (1) of Theorem 3.1.

Lemma 3.2 *Statement (1) of Theorem 3.1 implies that for some* $\mathbb{B}_\theta$*-name* τ, *we have* $supt(\tau) \subseteq \omega$, $1 \Vdash_{\mathbb{B}_\theta} \tau : \omega \to 2$, *and* $1 \Vdash_{\mathbb{B}_\theta} \ (\forall Y \subseteq \theta)\ \varphi(\tau, Y)$*.*

Proof: Apply the maximal principle to get such a τ; we can always permute the coordinates to make $supt(\tau) \subseteq \omega$. ∎

In the case that the τ here happens to be the official random real $\dot{r}$, Theorem 3.1 is immediate from the following:

Lemma 3.3 *Fix* $\theta \geq \omega_1$ *such that each uncountable* $\alpha \leq \theta$ *is not rveee, and assume* ME_θ*. Let* ψ *be a first order formula over a* θ*-structure* $\mathfrak{A}$*. Let* $Y \subseteq 2^\omega \times \theta$*. Assume it is true (in* V*) that*

$$(\forall x : \omega \to 2)\ \neg\psi(x, Y_x) \quad . \tag{2'}$$

Then for some $\mathbb{B}_\theta$ *name* $\dot{Z}$, $1 \Vdash_{\mathbb{B}_\theta} \dot{Z} \subseteq \theta$ *and*

$$1 \Vdash_{\mathbb{B}_\theta} \ \neg\psi(\dot{r}, \dot{Z}) \quad . \tag{1'}$$

Proof: If ν is any measure on 2^ω extending Lebesgue measure such that each Y^α is ν-measurable, we may define the $\mathbb{B}(\nu)$-name $\dot{Y} = \{\langle \check{\alpha}, [Y^\alpha] \rangle : \alpha < \theta\}$. Thus, $1 \Vdash_{\mathbb{B}(\nu)} \dot{Y} \subseteq \theta$ and each $\|\alpha \in \dot{Y}\| = [Y^\alpha]$. As a preliminary, we shall establish

$$1 \Vdash_{\mathbb{B}(\nu)} \ \neg\psi(\dot{r}, \dot{Y}) \quad . \tag{1''}$$

To conclude this from (2′), we need to measure not only the Y^α, but some sets derived from the formula ψ as well.

To simplify the argument, we shall measure more sets than necessary. As usual ([4]), let $\mathcal{L}_A$ be the language of $\mathfrak{A}$ augmented by a constant symbol $\ulcorner\alpha\urcorner$

for each element α of θ, the domain of $\mathfrak{A}$. For each sentence χ of $\mathcal{L}_A$, let $E_\chi = \{x \in 2^\omega : \chi(x, Y_x)\}$. Let $\mathcal{F}$ be the collection of all the E_χ, where χ is a sentence of $\mathcal{L}_A$. Note that $\mathcal{F}$ has size θ and includes all the Y^α (by using atomic χ).

To prove ($1''$) from ($2'$), we shall produce a measure algebra in which we have $\|\chi(\dot{r}, \dot{Y})\| = [E_\chi]$ for any sentence χ of $\mathcal{L}_A$. The proof of $\|\chi(\dot{r}, \dot{Y})\| = [E_\chi]$ will be by induction on the complexity of χ, but for this proof to work, the measure algebra will have to preserve certain suprema. In particular, each induction step handling an existential quantifier gives us a supremum to preserve: For each formula $\chi(v)$ of $\mathcal{L}_A$ with one free variable v, let $\mathcal{E}_\chi = \{E_{\chi(\ulcorner\alpha\urcorner)} : \alpha < \theta\}$. Note that $E_{\exists v \chi(v)} = \bigcup_{\alpha<\theta} E_{\chi(\ulcorner\alpha\urcorner)}$, so that a measure algebra satisfies

$$[E_{\exists v \chi(v)}] = \bigvee_{\alpha<\theta} [E_{\chi(\ulcorner\alpha\urcorner)}] \tag{$*$}$$

iff it preserves $\bigvee \mathcal{E}_\chi$. Apply Theorem 2.8 to choose $\mathcal{S} \subseteq \mathcal{P}(2^\omega)$ of size θ such that each measure on 2^ω which measure $\mathcal{F} \cup \mathcal{S}$ preserves each $\bigvee \mathcal{E}_\chi$. Then apply ME_θ to fix a measure ν, extending Lebesgue measure, which measures $\mathcal{F} \cup \mathcal{S}$.

Now, in the measure algebra $\mathbb{B}(\nu)$, induction on the complexity of χ does indeed show that $\|\chi(\dot{r}, \dot{Y})\| = [E_\chi]$ for each sentence χ of $\mathcal{L}_A$. Then, in particular, for the sentence $\neg\psi$, ($2'$) says that $E_{\neg\psi} = 2^\omega$, so that $\|\neg\psi(\dot{r}, \dot{Y})\| = 1$, proving ($1''$).

Finally, we conclude ($1'$) by Maharam's Theorem. We may assume that ν was chosen so that the ν-measurable sets are generated by the θ sets in $\mathcal{F} \cup \mathcal{S}$ together with the basic clopen sets in 2^ω, so that the Maharam dimension of ν cannot exceed θ. Thus, there is an isomorphism i from $\mathbb{B}(\nu)$ onto some complete subalgebra $\mathbb{D}$ of $\mathbb{B}_\theta$, with $i([\{f : f(n) = \ell\}]_{\mathbb{B}(\nu)}) = [\{f : f(n) = \ell\}]_{\mathbb{B}_\theta}$ for each $n < \omega$ and $\ell < 2$. Since i is an isomorphism, $1 \Vdash_{\mathbb{D}} \neg\psi(\dot{r}, \dot{Z})$, where $\dot{Z}$ is the corresponding $\mathbb{D}$-name; that is $\|\alpha \in \mathcal{D}\| = i(\|\alpha \in \dot{Y}\|)$. Since ψ is first order, it is absolute, so ($1'$) follows. ∎

To finish the proof of Theorem 3.1, we must deal with the possibility that the τ resulting from Lemma 3.2 is different from $\dot{r}$. To do this, we use the fact that τ may be constructed from $\dot{r}$ by countable operations. Specifically, suppose $\sigma(x, v)$ is an $\mathcal{L}$ formula; that is, σ mentions the parameter x, and has a free first order variable v, but does not mention Y. Then, given the

θ-structure $\mathfrak{A}$, and $x \in 2^\omega$, define $z_\sigma^{\mathfrak{A}}(x) \in 2^\omega$ by

$$z_\sigma^{\mathfrak{A}}(x)(n) = 1 \Leftrightarrow \mathfrak{A} \models \sigma(x, n) \ .$$

Lemma 3.4 *If τ is a $\mathbb{B}_\theta$-name, supt$(\tau) \subseteq \omega$, and $1 \Vdash \tau : \omega \to 2$, then (in V) there are a θ-structure $\mathfrak{A}$ for some countable $\mathcal{L}$ and an $\mathcal{L}$ formula σ such that $1 \Vdash \tau = z_\sigma^{\mathfrak{A}}(\dot{r})$.*

Actually, σ need only quantify over ω, so that the map $x \mapsto z_\sigma^{\mathfrak{A}}(x)$ is a Borel map from 2^ω to 2^ω. The $\mathcal{L}$ and the $\mathfrak{A}$ from this lemma have nothing at all to do with the $\mathcal{L}$ and the $\mathfrak{A}$ from Theorem 3.1. Nevertheless, by merging the two structures, we may assume that they are the same.

Proof of Theorem 3.1: Assume (1), and let τ be as in Lemma 3.2. Then, let σ be as in Lemma 3.4. Thus, $1 \Vdash_{B_\theta} (\forall Y \subseteq \theta)\ \psi(\dot{r}, Y)$, where $\psi(x, Y)$ is the formula which asserts $\varphi(z_\sigma(x), Y)$. Now, applying Lemma 3.3, we get that condition $(1')$ must be false for every name $\dot{Z}$, so that we could never have chosen a $Y \subseteq 2^\omega \times \theta$ to satisfy $(2')$; hence

$$(\exists x : \omega \to 2)(\forall Y \subseteq \theta)\ \varphi(z_\sigma^{\mathfrak{A}}(x), Y_x) \quad ,$$

which implies (2). ∎

We now explain how to derive Theorem 1.3 from our absoluteness result. We concentrate on 1.3(1), the non-productivity of the *ccc*, since the three parts to Theorem 1.3 are similar.

The following standard construction produces a pair of partial orders whose product is never *ccc*. Start with a graph $\Phi : [\omega_1]^2 \to 2$ on ω_1 vertices. Define the partial orders $\mathbb{P}_0^\Phi$ and $\mathbb{P}_1^\Phi$ by setting $\mathbb{P}_\ell^\Phi = \{a \in [\omega_1]^{<\omega} : a$ is homogeneous for color $\ell\}$, and ordering each $\mathbb{P}_\ell^\Phi$ by reverse inclusion. Clearly, the product $\mathbb{P}_0^\Phi \times \mathbb{P}_1^\Phi$ isn't *ccc*: consider the subset $\{\langle \{\alpha\}, \{\alpha\} \rangle : \alpha < \omega_1\}$. Galvin [9] showed that under CH there is a graph Φ such that $\mathbb{P}_0^\Phi$ and $\mathbb{P}_1^\Phi$ are both *ccc*. By a different argument, Roitman [20] showed how to read off such a Φ from a random real. We use Roitman's construction here:

Proof of Theorem 1.3(1): Following [20], fix (in V) injective functions $f_\alpha : \alpha \to \omega$ for each $\alpha < \omega_1$.

For (1), given any $x : \omega \to 2$, we define the graph $\Phi(x) : [\omega_1]^2 \to 2$ by $\Phi(x)(\{\alpha, \beta\}) = x(f_\beta(\alpha))$ for each $\alpha < \beta < \omega_1$. By [20], 1 forces that each $\mathbb{P}^{\Phi_\ell(\dot{r})}$ is *ccc*; actually, [20] just states this for the forcing which adds only the one random real $\dot{r}$, but then it must be true also for the $\mathbb{B}_\theta$ extension, since random real forcing never destroys the *ccc*. Now, we can code the f_α along with the construction of the $\Phi(x)$ in a suitable θ-structure $\mathfrak{A}$, and apply Theorem 3.1 with $\theta = \omega_1$. The formula $\varphi(x, Y)$ says that either Y is bounded in ω_1 or Y fails to be an antichain in one of the $\mathbb{P}^{\Phi_\ell(\dot{r})}$. ∎

In [20], Roitman also used similar techniques to get S-spaces and L-spaces from a single random real, and this implies Theorem 1.3(2); we omit the details, which are almost verbatim the same as for Theorem 1.3(1). Note, however, that to prove Theorem 1.3(2) we cannot simply quote the fact that adding a random real adds an S-space and an L-space; we use the fact that the proof constructs these spaces in a first-order way from the random real and some structure in the ground model. In addition, using our absoluteness result to prove Theorem 1.3(2) requires the fact that random real forcing never destroys S-spaces and L-spaces. Similarly, Todorčević' construction of an entangled set (see page 55 of [25]) gives us Theorem 1.3(3). Actually, he shows that adding one random real adds an entangled set of size continuum. Thus, we get:

Theorem 3.5 *Assume ME_θ, where $\theta \geq \omega_1$ and no uncountable $\alpha \leq \theta$ is rveee. Then there is an $\aleph_1$-entangled set of size θ.*

4 Ramsey Ultrafilters.

We begin by explaining the property of Ramsey ultrafilters we intend to refute. The following fact is easy to see, and was used also in [14].

Lemma 4.1 *Suppose that $\mathcal{U}$ is a Ramsey ultrafilter on ω.*

1. *Given $a_i \in [0,1]$ for $i \in \omega$, there is an $H \in \mathcal{U}$ such that the sequence $\langle a_i : i \in H\rangle$ converges.*

2. *Given $a_i \in [0,1]$ for $i \in \omega$ with $\lim_{i\to\infty} a_i = 0$, there is an $H \in \mathcal{U}$ such that $\sum_{i\in H} a_i \leq 1$.*

Of course, in (2), once the sum is finite, it may be made arbitrarily small by choosing a smaller H.

Definition 4.1 *A nonprincipal ultrafilter $\mathcal{U}$ on ω is* semi-selective *iff $\mathcal{U}$ satisfies the conclusions (1) and (2) to Lemma 4.1.*

This notion has occurred with different names in the literature. The term "semi-selective" is taken from [14], where it was pointed out that under CH (or MA), there are semi-selective ultrafilters which are not Ramsey. We shall show that there are no semi-selective ultrafilters if either ME_ω holds or if the universe was obtained by adding at least ω_2 random reals to a model of ZFC.

We remark that, taken separately, conclusions (1) and (2) of Lemma 4.1 yield two weaker properties of ultrafilters, neither of which can be refuted in random real models (or, from any ME_θ). A nonprincipal ultrafilter satisfying property (2) is sometimes called "rapid". There are none of these in the Laver model (Miller [18]), but when we add random reals, any extension of a rapid ultrafilter from the ground model will still be a rapid ultrafilter. Satisfying property (1) is equivalent to being a P-point, and by a result of P. E. Cohen [5] there are P-points in every random real extension of a model of CH.

By amalgamating properties (1) and (2), we get the following lemma.

Lemma 4.2 *Let f be any continuous real-valued function on $[0,1]$ such that $f(0) = 0$, and $f(x) > 0$ for $x > 0$. Let $\mathcal{U}$ be any nonprincipal ultrafilter on ω. Then $\mathcal{U}$ is semi-selective iff for all $\Phi : \omega \to [0,1]$, there are $H \in \mathcal{U}$ and $t \in [0,1]$ such that*

$$\sum_{i \in H} f(|\Phi(i) - t|) \leq 1 \qquad (*)$$

Of course, $(*)$ implies $\lim_{i \in H} \Phi(i) = t$, since H is infinite. Now, consider f to be fixed. Just in ZFC, there is no problem choosing, for each $\Phi \in [0,1]^\omega$, an infinite H_Φ satisfying $(*)$. We now present a probabilistic argument showing that the set of all these H_Φ can never have the finite intersection property, so that there can be no semi-selective ultrafilters. Of course, this argument only works in some models of set theory.

The intuition is: Choose $\Phi \in [0,1]^\omega$ at random, and then, by some (non-random) process, choose $H_\Phi \subseteq \omega$ and $t_\Phi \in [0,1]$ so that $(*)$ holds. If f is "really large", then each H_Φ must be so thin that with probability 1, we will have chosen H_{Φ_1} and H_{Φ_2} such that $H_{\Phi_1} \cap H_{\Phi_2}$ is finite.

Corollary 4.4 below formalizes this intuition. First, some notation. Let us use λ both for Lebesgue measure on $[0,1]$, as well as for the usual product measure on $[0,1]^I$ for any set I. Let μ be a probability measure on a set X. A *random I-sequence* (indexed by the sample space (X,μ)) is a map $\Psi : X \to [0,1]^I$ such that for each Baire set $B \subseteq [0,1]^I$, the set $\mu(\{x : \Psi_x \in B\})$ is μ-measurable and of measure $\lambda(B)$. Note that we are using Ψ_x for $\Psi(x)$.

Suppose $H \subseteq X \times I$. We let $H_x = \{i \in I : (x,i) \in H\}$ and $H^i = \{x \in X : (x,i) \in H\}$.

For the rest of this section, let $f(x) = -100/(\ln(x) - 2)$ for $0 < x \leq 1$, and let $f(0) = 0$. The justification for using this particular f is only that it makes Theorem 4.3 true.

Given any $\Phi \in [0,1]^I$, we say that $H \subseteq I$ is Φ - *small* iff for some $t \in [0,1]$, $\sum_{i \in H} f(|\Phi(i) - t|) \leq 1$. A *small process* for a random I-sequence Ψ is a set $H \subseteq X \times I$ such that for each $x \in X$, H_x is Ψ_x - small, and for each $i \in I$, H^i is μ-measurable. Trivial examples of small processes are $\emptyset$, or $X \times \{i\}$ for any $i \in I$.

Theorem 4.3 *For any set I, if H is a small process for a random I - sequence, then $\sum_{i \in I} (\mu(H^i))^2 \leq 1.9$.*

Assuming Theorem 4.3 for a moment, we prove the following two corollaries; the second one immediately implies Theorem 1.4.

Corollary 4.4 *For any countable set I, if H is a small process for a random I - sequence, then there are some x, y such that $|H_x \cap H_y| \leq 1$*

Proof: Computing the expectation of $|H_x \cap H_y|$, we have, by Theorem 4.3,

$$\int dx \int dy\ |H_x \cap H_y| = \int dx \int dy \sum_{i \in \omega} \chi_H(x,i) \cdot \chi_H(y,i) =$$
$$\sum_{i \in \omega} \int dx \int dy\ \chi_H(x,i) \cdot \chi_H(y,i) = \sum_{i \in \omega} (\mu(H^i))^2 \leq 1.9$$

so there must be some x, y such that $|H_x \cap H_y| \leq 1$. ∎

Corollary 4.5 *ME_ω implies that there are no semi-selective ultrafilters.*

Proof: Let $\mathcal{U}$ be a semi-selective ultrafilter on ω. Let X be $[0,1]^\omega$ with the usual product measure. Let $\Psi_x = x$. For each x, choose $H_x \in \mathcal{U}$ such that H_x is Ψ_x-small. This defines $H \subseteq X \times \omega$. Applying ME_ω, let μ extend the usual product measure on X and measure all the H^i. Then Corollary 4.4 yields an immediate contradiction. ∎

Later, in proving Theorem 4.9, it will be important that Theorem 4.3 was stated for X an arbitrary sample space, not just $[0,1]^\omega$. Note that Corollary 4.4 has a non-vacuous content just in ZFC, since it is easy to find Borel small processes $H \subseteq [0,1]^\omega \times \omega$ such that each H_x is infinite.

We turn now to a proof of Theorem 4.3. We first consider finite products. For each $\Phi \in [0,1]^n$, let $h(\Phi)$ be the largest size of a Φ - small set. Note that h is Borel measurable. For $n \geq 0$, let σ_n be the expected value of $h(\Phi)$:

$$\sigma_n = \int h(\Phi)\, d\lambda(\Phi)$$

It is easy to see that $\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots$. Also, $\sigma_0 = 0$ and $\sigma_1 = 1$, since every set of size 1 is small. Also, $\sigma_n \nearrow \infty$ as $n \nearrow \infty$, although the growth rate is fairly small, as we shall show in Lemma 4.7. First, we use the size of the σ_n to place a crude upper bound to the sum in Theorem 4.3.

Lemma 4.6 *Let I be any set, and let H be a small process for a random I - sequence. Then $\sum_{i\in I}(\mu(H^i))^2 \leq \sum_{m=1}^{\infty}(\sigma_m/m)^2$.*

Proof. It suffices to prove this for I finite, and it is trivial if I is empty, so say $|I| = n+1$. Then, we may as well assume that I is the ordinal $n+1$, arranged so that $\mu(H^0) \geq \mu(H^1) \geq \cdots \geq \mu(H^n)$. For each $m \leq n+1$, $H_x \cap m$ is $\Psi_x \restriction m$ - small, so $|H_x \cap m| \leq h(\Psi_x \restriction m)$. Then

$$\sum_{i<m} \mu(H^i) = \int_X |H_x \cap m|\, d\mu(x) \leq \int_X h(\Psi_x \restriction m)\, d\mu(x) =$$

$$\int_{[0,1]^m} h(\Phi)\, d\lambda(\Phi) = \sigma_m$$

Setting $m = j+1$,

$$\mu(H^j) \leq \frac{1}{m}\sum_{i<m}\mu(H^i) \leq \sigma_m/m$$

(since $\mu(H^i) \searrow$ as $i \nearrow$), and the result now follows by summing over j. ∎

Lemma 4.7 *Whenever* $1 \le r \le n$, $\sigma_n \le (2r-1) + n^3 e^2 \cdot e^{-100r}$.

Proof. First note that f' is positive and f'' is negative on $(0,1)$, which implies that f is increasing on $[0,1]$ and $f(a+b) \le f(a) + f(b)$ whenever $0 \le a, b \le a+b \le 1$. Also, the inverse of f is a "small" exponential. If $y = -100/(\ln(x) - 2)$ then $x = e^2 \cdot e^{-100/y}$.

Now, fix n, r, and let $S = \{\Phi \in [0,1]^n : h(\Phi) \ge 2r\}$. Then, $\sigma_n \le (2r-1) + n\lambda(S)$. To estimate $\lambda(S)$, fix $\Phi \in S$ and then fix $H \subseteq n$ such that $|H| = 2r$ and H is Φ-small. Then, fix $t \in [0,1]$ such that $\sum_{i \in H} f(|\Phi(i) - t|) \le 1$. List H as $\{i_1, j_1, \ldots, i_r, j_r\}$. For $\ell = 1, \ldots, r$, $f(|\Phi(i_\ell) - \Phi(j_\ell)|) \le f(|\Phi(i_\ell) - t|) + f(|\Phi(j_\ell) - t|)$. Since the sum is ≤ 1, there is an ℓ such that $f(|\Phi(i_\ell) - \Phi(j_\ell)|) \le 1/r$. So, we have shown that for each $\Phi \in S$, there are distinct i, j such that $f(|\Phi(i) - \Phi(j)|) \le 1/r$.

Now, for each particular $i \ne j$, $\lambda\{\Phi \in [0,1]^n : f(|\Phi(i) - \Phi(j)|) \le 1/r\} \le 2e^2 \cdot e^{-100r}$. Since there are $\binom{n}{2} \le n^2/2$ possibilities for i, j, we have $\lambda(S) \le n^2 e^2 \cdot e^{-100r}$, proving the lemma. ∎

Of course, for some r, n, the estimate in Lemma 4.7 is worse than the obvious $\sigma_n \le n$, but by choosing an appropriate r for each n, we may obtain an upper bound which is sufficient to prove the desired result.

Proof of Theorem 4.3: By Lemma 4.6, it is sufficient to prove that $\sum_{n=1}^{\infty} (\sigma_n/n)^2 \le 1.9$. To do this we break the sum into blocks, where block 1 sums from $n = 1$ to 100 and block r, for $r > 1$ sums for $n = 100^{r-1} + 1$ to 100^r. On each block, we apply Lemma 4.7.

On block 1: For $n = 1, \ldots, 100$, we have $\sigma_n \le 1 + 10^6 e^2 \cdot e^{-100} \le 1.01$ (using $r = 1$), which implies that

$$\sum_{n=1}^{100} (\sigma_n/n)^2 \le \frac{\pi^2}{6} \cdot 1.01^2 \le 1.7$$

On block r, for $r > 1$: $\sigma_n \le 2r$, so

$$\sum_{n=100^{r-1}+1}^{100^r} (\sigma_n/n)^2 \le (2r)^2 \sum_{n=100^{r-1}+1}^{\infty} \frac{1}{n^2} \le \frac{(2r)^2}{100^{r-1}}$$

and it is easy to see that

$$\sum_{r=2}^{\infty} \frac{(2r)^2}{100^{r-1}} \le 0.2$$

∎

We now add forcing to this proof and show that there are no semi-selective ultrafilters in any model obtained by adding ω_2 or more random reals. We continue the notation of §§2,3, forcing with $\mathbb{B}_\kappa$. We continue to use λ to denote Lebesgue measure on $[0,1]$ or any power thereof, and μ to denote the usual measure on any power of $2 = \{0,1\}$.

Definition 4.2 *For any set I, a* random I-sequence-name *is a name π in the forcing language such that $\|\pi : I \to [0,1]\| = 1$, and for each Baire $B \subseteq [0,1]^I$, $\mu(\|\pi \in B\|) = \lambda(B)$.*

Note that in the formalism of forcing, inside the $\|\cdots\|$, I should really be $\check{I}$, whereas B should really be a Borel code (see [17]), *not* $\check{B}$.

Lemma 4.8 *Suppose that π_1 and π_2 are random I-sequence-names and suppose that τ_1 and τ_2 names in the forcing language such that each τ_ℓ is forced by 1 to be a subset of I and to be π_ℓ-small ($\ell = 1,2$). Assume that $supt(\tau_1) \cap supt(\tau_2)$ is disjoint from $supt(\pi_1) \cup supt(\pi_2)$. Then some forcing condition p forces $|\tau_1 \cap \tau_2| \leq 1$.*

Proof. This lemma involves forcing, so it may be perceived to take place in some ground model, V. The proof is more transparent if we view V as being countable from the outside, so we may refer explicitly to generic objects. Let $Z = 2^\kappa$. In the case of random real forcing, it is more convenient to think of the generic object as an object $z \in Z$ which is *random over V* (that is, not in any V-coded Baire nullset), rather than a generic filter (see [17]). To prove the lemma, it is sufficient to find some z random over V such that $|val(\tau_1, z) \cap val(\tau_2, z)| \leq 1$ (where val refers to the value of a name in the generic extension).

Let $T = supt(\tau_1) \cap supt(\tau_2)$, let $Y = 2^T$, and let $X = 2^{\kappa \setminus T}$. Then we may identify Z with $X \times Y$, and think of the extension $V[z] = V[(x,y)]$ as the iterated extension $V[y][x]$. Fix a $y \in Y$ which is random over V. Then for all $x \in X$, x is random over $V[y]$ iff (x,y) is random over V. Furthermore, x is random over $V[y]$ for almost every $x \in X$. For $\ell = 1,2$, let $\Psi_{\ell,x} = val(\pi_\ell, (x,y))$. Since $supt(\pi_\ell)$ is disjoint from T, Ψ_ℓ is a random I-sequence. Define $H_\ell \subseteq X \times I$ so that $H_{\ell,x} = val(\tau_l, (x,y))$ when x is random over $V[y]$, and $H_{\ell,x} = \emptyset$ otherwise. Then H_ℓ is a small process for Ψ_ℓ. For each $i \in I$, $supt(\|i \in \tau_1\|) \cap supt(\|i \in \tau_2\|) \subseteq T$, which implies that H_1^i and H_2^i are (stochastically) independent. Now, applying Theorem 4.3 and the Cauchy-Schwarz inequality:

$$\int |H_{1,x} \cap H_{2,x}| dx = \sum_{i \in I} \mu(H_1^i \cap H_2^i) = \sum_{i \in I} \mu(H_1^i) \cdot \mu(H_2^i) \leq$$
$$(\sum_{i \in I} \mu(H_1^i)^2)^{1/2} \cdot (\sum_{i \in I} \mu(H_2^i)^2)^{1/2} \leq 1.9$$

Hence, $\mu\{x \in X : |H_{1,x} \cap H_{2,x}| \leq 1\} \geq 0.1$, so we may choose an x random over $V[y]$ such that $val(\tau_1, (x, y)) \cap val(\tau_2, (x, y)) = H_{1,x} \cap H_{2,x}$ has size 0 or 1, as required. ∎

Now, Theorem 1.5 follows immediately from:

Theorem 4.9 *Suppose that $\kappa \geq \omega_2$. Then it is forced by* 1 *that there are no semi-selective ultrafilters in the generic extension by $\mathbb{B}_\kappa$.*

Proof. If the theorem fails, then by homogeneity of $\mathbb{B}_\kappa$, there would be a name $\dot{\mathcal{U}}$ such that 1 forces $\dot{\mathcal{U}}$ to be a Ramsey ultrafilter. In V, fix disjointly supported random ω-sequence-names, π_α, for $\alpha < \omega_2$. Then fix names τ_α such that 1 forces that $\tau_\alpha \in \dot{\mathcal{U}}$ and that τ_α is π_α-small. By the usual pressing-down argument, we may find distinct α, β such that $supt(\tau_\alpha) \cap supt(\tau_\beta)$ is disjoint from $supt(\pi_\alpha) \cup supt(\pi_\beta)$. Then $\tau_\alpha, \tau_\beta, \pi_\alpha, \pi_\beta$ contradict Lemma 4.8. ∎

5 Random Graphs

In this section, we show how to derive Theorems 1.4 and 1.5 directly from a result in random graph theory about partitions on finite sets. For background in this subject, see [1][23].

The intuition is: for a fixed set I, choose a partition $\Psi : [I]^2 \to 2$ at random, and then, by some (non-random) process, construct a homogeneous set H for Ψ. There are many results in the literature, going back to a 1947 paper of Erdös [6], to the effect that with high probability, H must be fairly "thin". Erdös used this to establish an exponential lower bound for the Ramsey numbers.

To formalize this intuition, we use the following general framework. Let (X, μ) be a probability space. A *random partition* of a set I, *indexed* by

(X, μ), is a map Ψ such that for each $x \in X$, Ψ_x (i.e., $\Psi(x)$) is a partition, $\Psi_x : [I]^2 \to 2$, and such that the sets $E_{i,j} = \{x : \Psi_x(i,j) = 0\}$, for $\{i,j\} \in [I]^2$, are μ-measurable and are independent events of probability (i.e., measure) $\frac{1}{2}$.

A *homogeneous process* (for Ψ) is a set $H \subseteq X \times I$ such that for each $x \in X$, H_x is homogeneous for Ψ_x, and for each $i \in I$, H^i is μ-measurable. Trivial examples of homogeneous processes are $\emptyset$, or $X \times \{i,j\}$ for any $i,j \in I$. The specific theorem we need is:

Theorem 5.1 *If H is a homogeneous process for a random partition of a set I, then $\sum_{i \in I} (\mu(H^i))^2 \leq 3.96$.*

Note that Theorem 5.1 does not assume I to be finite, although once we prove it for finite I, it follows immediately for all I. From Theorem 5.1, we may derive Theorems 1.4 and 1.5 exactly as in §4; we omit the details of this.

Before proving Theorem 5.1, we mention the following corollary in finite combinatorics. Roughly, if we choose homogeneous sets for partitions using any (non-random) process, then there is a non-0 probability that two of the homogeneous sets will have small intersection. Formally,

Corollary 5.2 *Assume that I is countable (or finite), Ψ is a random partition of I indexed by (X, μ), and H is a homogeneous process for Ψ. Then* $\mu \times \mu(\{(x,y) : |H_x \cap H_y| \leq 3\}) \geq 0.01$

Proof: Let $E = \int dx \int dy \; |H_x \cap H_y|$ (that is, the expectation of $|H_x \cap H_y|$). Let $w = \mu \times \mu(\{(x,y) : |H_x \cap H_y| \leq 3\})$. Then $E \geq 4 \cdot (1 - w)$. But also, exactly as in the proof of Corollary 4.4, we have $E \leq 3.96$. Combining these, we get $w \geq 0.01$. ∎

We remark that the corollary is probably interesting primarily for the special case of the "natural" random partition, where X is $2^{[I]^2}$ (with the usual product measure), and Ψ_x is just x, but we seem to need the more general statement of Theorem 5.1 to derive Theorem 1.5, and the proof of the general statement is no harder than the proof of the special case.

We proceed now to prove Theorem 5.1. Actually, we suspect that our result is not best possible, in that possibly the "3.96" in Theorem 5.1 could be replace by "2.97", which would mean that the "≤ 3" in Corollary 5.2 could be replaced by "≤ 2". However, our proof involves a sequence of estimates, each one introducing a bit of slop in the final result.

First, the following lemma can be used to bound a sum of squares:

Lemma 5.3 *Suppose $a_0, \ldots, a_n$, $b_0, \ldots, b_n$ are real numbers such that*

$$a_0 + \cdots + a_j \leq b_0 + \cdots + b_j \text{ for } j = 0, \ldots, n; \quad \text{and} \quad a_0 \geq a_1 \geq \cdots \geq a_n \geq 0 \quad (*)$$

Then $a_0^2 + \cdots + a_n^2 \leq b_0^2 + \cdots + b_n^2$.

Proof: We may assume that each $b_j \geq 0$ (otherwise replace b_j by $|b_j|$). Since the lemma is trivial for $n = 0$, we proceed by induction. So, fix $n > 0$, and fix non-negative b_j, and assume that the lemma holds for all smaller values of n. Now, by compactness, fix numbers $a_0, \ldots, a_n$ satisfying $(*)$ which maximize $a_0^2 + \cdots + a_n^2$.

If, for some $j < n$, we have $a_0 + \cdots + a_j = b_0 + \cdots + b_j$, then we have $a_{j+1} + \cdots + a_k \leq b_{j+1} + \cdots + b_k$ for $k = j+1, \ldots, n$, so applying the induction hypothesis, $a_{j+1}^2 + \cdots + a_n^2 \leq b_{j+1}^2 + \cdots + b_n^2$. Since the induction hypothesis also implies that $a_0^2 + \cdots + a_j^2 \leq b_0^2 + \cdots + b_j^2$, we have $a_0^2 + \cdots + a_n^2 \leq b_0^2 + \cdots + b_n^2$.

So, assume that $a_0 + \cdots + a_j < b_0 + \cdots + b_j$ for each $j < n$. We must then have $a_0 + \cdots + a_n = b_0 + \cdots + b_n$; otherwise, we could replace each a_i by some $a_i + \epsilon$ and contradict maximality. But then $a_n > 0$, so, for a small enough ϵ, the sequence $a_0 + \epsilon, a_2, \ldots, a_{n-1}, a_n - \epsilon$ satisfies $(*)$ and contradicts maximality, since $a_0 \geq a_n$ implies that $(a_0 + \epsilon)^2 + (a_n - \epsilon)^2 > a_0^2 + a_n^2$. ∎

We shall prove Theorem 5.1 by using this lemma plus a crude upper bound (Lemma 5.4) on the partial sums of the $\mu(H_i)$. Let $\mathcal{G}_n = 2^{([n]^2)}$, and let ν denote the usual counting probability measure on $\mathcal{G}_n$. We may think of elements of $\mathcal{G}_n$ as random graphs (or partitions) on n nodes, since each $\Phi \in \mathcal{G}_n$ is a partition of the pairs from the n-element set $n = \{0, \ldots, n-1\}$ into 2 pieces. For each $\Phi \in \mathcal{G}_n$, let $h(\Phi)$ be the largest size of a homogeneous set for Φ. For $n \geq 0$, let σ_n be the expected value of $h(\Phi)$:

$$\sigma_n = \int h(\Phi)\, d\nu(\Phi)$$

It is easy to see that $\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots$. Also, $\sigma_0 = 0$, $\sigma_1 = 1$ and $\sigma_2 = 2$, since every set of size 2 or less is homogeneous. It is not hard to see by direct computation (Lemma 5.5) that $\sigma_3 = 2.25$ and $\sigma_4 = 2.75$. It is well-known

[1][23] that for large n, σ_n is approximately $2 \cdot lg(n)$, where $lg(x)$ is $log_2(x)$. A suitable upper bound on the σ_n can be used to prove Theorem 5.1 by applying the following lemma:

Lemma 5.4 *Suppose that for each m, $s_m \geq \sigma_m$, with $s_0 = 0$. As in Theorem 5.1, let H be a homogeneous process for a random partition Ψ of a set I. Then* $\sum_{i \in I} (\mu(H^i))^2 \leq \sum_{n=0}^{\infty} (s_{n+1} - s_n)^2$.

Proof: As remarked above, it is sufficient to prove this when I is finite, and it is trivial if I is empty, so say $|I| = n+1$. Then, we may as well assume that I is the ordinal $n+1$, arranged so that $\mu(H^0) \geq \mu(H^1) \geq \cdots \geq \mu(H^n)$. For each $m \leq n+1$, $H_x \cap m$ is homogeneous for $\Psi_x \restriction [m]^2$, so $|H_x \cap m| \leq h(\Psi_x \restriction [m]^2)$. Then

$$\sum_{i<m} \mu(H^i) = \int_X |H_x \cap m| \, d\mu(x) \leq \int_X h(\Psi_x \restriction [m]^2) \, d\mu(x) =$$
$$\int_{\mathcal{G}_m} h(\Phi) \, d\nu(\Phi) = \sigma_m \leq s_m$$

Let $a_i = \mu(H^i)$ and $b_i = s_{i+1} - s_i$. Setting $m = j+1$, for $j = 0, \ldots, n$, we have $a_0 + \cdots + a_j \leq s_{j+1} = b_0 + \cdots + b_j$ (since $s_0 = 0$), so the result follows by Lemma 5.3 ∎

Using the approximation $\sigma_n \approx 2 \cdot lg(n)$, together with the fact that $(log(n+1) - log(n)) = log(1 + \frac{1}{n}) \leq \frac{1}{n}$, we can now bound $\sum_{i \in I} (\mu(H^i))^2$ by something like $(\frac{2}{log(2)})^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \approx 13.69$. However, to achieve the bound of 3.96, we must work a little harder. For small n, we use the exact value σ_n, rather than the asymptotic approximation.

Lemma 5.5 $\sigma_0 = 0$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 2.25$, $\sigma_4 = 2.75$.

Proof: For σ_3: $\mathcal{G}_3$ has size $2^3 = 8$. By inspection, 2 of the $\Phi \in \mathcal{G}_3$ have $h(\Phi) = 3$, and 6 of the $\Phi \in \mathcal{G}_3$ have $h(\Phi) = 2$.

For σ_4: $\mathcal{G}_4$ has size $2^6 = 64$. By inspection, 2 of the $\Phi \in \mathcal{G}_4$ have $h(\Phi) = 4$, 44 of the $\Phi \in \mathcal{G}_4$ have $h(\Phi) = 3$, and 18 of the $\Phi \in \mathcal{G}_3$ have $h(\Phi) = 2$. ∎

For larger n, we compute an upper bound on σ_n as follows. For $r \leq n$, we define $\epsilon_r^n = \nu\{\Phi \in \mathcal{G}_n : h(\Phi) \geq r\}$.

Lemma 5.6 *a. For $r < n$:*

$$\sigma_n \leq r - 1 + \epsilon_r^n + \epsilon_{r+1}^n \cdot (n - r)$$

b. For $r \leq n$:

$$\epsilon_r^n \leq \binom{n}{r} \cdot 2^{1+r/2-r^2/2}$$

Proof: For (a), we partition $\mathcal{G}_n$ into three pieces, where $h(\Phi) < r$, $h(\Phi) = r$, and $h(\Phi) > r$. This yields $\sigma_n \leq (r-1) \cdot (1 - \epsilon_r^n) + r \cdot (\epsilon_r^n - \epsilon_{r+1}^n) + n \cdot \epsilon_{r+1}^n$.

For (b), note that $h(\Phi) \geq r$ iff there is some $A \subseteq n$ of size r which is homogeneous for Φ. Assume $r \geq 2$, since otherwise (b) is trivial (since $\epsilon_r^n \leq 1$). Then, there are exactly $\binom{n}{r}$ possibilities for A, and each A can be homogeneous by having all its pairs colored either 0 or 1, so the probability that A is homogeneous is exactly $2 \cdot 2^{-\binom{r}{2}}$. ∎

These estimates on σ_n are not quite as crude as they may seem at first sight, since it is known [1][23] that for large n, there is some $r \approx 2 \cdot lg(n)$ such that "most" of the $\Phi \in \mathcal{G}_n$ have $h(\Phi)$ equal to r or $r-1$, so that $\sigma_n \approx r$. For this r, the estimates in Lemma 5.6 will compute $\epsilon_{r+1}^n \cdot n$ to be negligible, bounding σ_n by some value near r. It was our use of σ_n at all, in Lemma 5.4, that was *really* crude.

Lemma 5.7 *$\sigma_1 \leq 1.0$, $\sigma_2 \leq 2.0$, $\sigma_3 \leq 2.375$, $\sigma_4 \leq 2.75$, $\sigma_5 \leq 3.159$, $\sigma_6 \leq 3.493$, $\sigma_7 \leq 4.042$, $\sigma_8 \leq 4.115$, $\sigma_9 \leq 4.267$, $\sigma_{10} \leq 4.557$, $\sigma_{11} \leq 5.030$, $\sigma_{12} \leq 5.061$.*

Proof: We just choose the value the r which yields the best estimate on σ_n by Lemma 5.6. For $n = 5, 6$, we use $r = 4$; for $n = 7, 8, 9, 10$, we use $r = 5$; for $n = 11, 12$, we use $r = 6$. For $n < 5$, we use Lemma 5.5 instead. ∎

Calling the estimate in Lemma 5.7 s_n (as in Lemma 5.4) (for $1 \leq n \leq 12$), and setting $s_0 = 0$, we get $\sum_{n=0}^{11}(s_{n+1} - s_n)^2 = 3.198711 \leq 3.199$. Note that replacing the exact 2.25 be the larger 2.375 yielded a smaller (by 1/32) value of the sum. Let $\delta = 2 \cdot lg(12) - s_{12} \approx 2.109$, so $s_{12} = 2 \cdot lg(12) - \delta$. To estimate the tail of the series, it will be sufficient to prove that $\sigma_n \leq 2 \cdot lg(n) - \delta$ for all $n \geq 12$. To do that, we use the following estimate.

Lemma 5.8 *Suppose that r is an integer with $11 \leq r < n$ and suppose $r = 2 \cdot lg(n) - a$, where $a \leq 3$. Then*

$$\sigma_n \leq r - 1 + (\frac{1}{4} + \frac{1}{r}) \cdot 2^{(a-3)r/2}$$

Proof: By Stirling's Formula, $r! > \sqrt{2\pi r} \cdot r^r e^{-r}$. Since $n = 2^{r/2+a/2}$ and $\binom{n}{r} \leq n^r/r!$, Lemma 5.6 implies

$$\epsilon_r^n \leq \frac{1}{\sqrt{2\pi r}} \cdot \left(\frac{e}{r}\right)^r \cdot 2^{1+r/2+ar/2}$$

Of course, the same estimate holds if we replace r by $r+1$ and a by $a-1$. We may simplify this by observing that $(1+1/r)^{r+1} \geq e$ for all $r > 1$ (take the *log* of both sides). It follows that $(e/(r+1))^{r+1} \leq e^r/(r^{r+1})$, so

$$\epsilon_{r+1}^n \leq \frac{1}{\sqrt{2\pi r}} \cdot \frac{1}{r} \cdot \left(\frac{e}{r}\right)^r \cdot 2^{3/2+ar/2}$$

Since $n = 2^{r/2+a/2}$,

$$\epsilon_{r+1}^n \cdot n \leq \frac{1}{\sqrt{2\pi r}} \cdot \frac{1}{r} \cdot \left(\frac{e}{r}\right)^r \cdot 2^{3/2+ar/2+r/2+a/2}$$

Since, by Lemma 5.6, $\sigma_n \leq r - 1 + \epsilon_r^n + \epsilon_{r+1}^n \cdot n$,

$$\sigma_n \leq r - 1 + \frac{1}{\sqrt{2\pi r}} \cdot \left(\frac{e}{r}\right)^r \cdot (1 + \frac{1}{r} \cdot 2^{1/2+a/2}) \cdot 2^{1+r/2+ar/2}$$

Now, $r \geq 11$ implies that $r \geq 4e$ and $2\pi r \geq 64$, so

$$\sigma_n \leq r - 1 + \frac{1}{4} \cdot (1 + \frac{1}{r} \cdot 2^{1/2+a/2}) \cdot 2^{-3r/2+ar/2}$$

Finally, using $a \leq 3$ yields the desired result. ∎

Lemma 5.9 *If $n \geq 128$ then $\sigma_n \leq 2 \cdot lg(n) - 2.5$.*

Proof: We apply Lemma 5.8. Choose a such that $2 \leq a \leq 3$ and such that $r = 2 \cdot lg(n) - a$ is an integer. Then $r - 1 \leq 2 \cdot lg(n) - 3$, and $\sigma_n \leq r - 1 + \frac{1}{4} + \frac{1}{4}$.
∎

Proof of Theorem 5.1: First note that for all $n \geq 13$, $\sigma_n \leq 2 \cdot lg(n) - 2.2$. For $n \geq 128$, this follows from Lemma 5.9. For smaller n, we just compute it using Lemma 5.6, using $r = 6$ for $13 \leq n \leq 17$, $r = 7$ for $18 \leq n \leq 26$, $r = 8$ for $27 \leq n \leq 40$, $r = 9$ for $41 \leq n \leq 62$, $r = 10$ for $63 \leq n \leq 95$, and $r = 11$ for $96 \leq n \leq 127$.

Thus, we can in fact set $s_n = 2 \cdot lg(n) - \delta$ for all $n \geq 12$, as indicated above, whence, applying Lemma 5.4

$$\begin{aligned}
\sum_{i \in I} (\mu(H^i))^2 &\leq \\
\sum_{n=0}^{11} (s_{n+1} - s_n)^2 + \sum_{n=12}^{\infty} (s_{n+1} - s_n)^2 &\leq \\
3.199 + (2/log(2))^2 \cdot \sum_{n=12}^{\infty} (log(n+1) - log(n))^2 &\leq \\
3.199 + (2/log(2))^2 \cdot \sum_{n=12}^{\infty} \frac{1}{n^2} &\leq \\
3.199 + (2/log(2))^2 \cdot \frac{1}{11} &< 3.96 \qquad \blacksquare
\end{aligned}$$

6 Additional Remarks.

We point out here that, at least for small θ, our results about ME_θ are best possible.

First, we note that in the axiom ME_ω, one cannot replace $[0,1]$ by an arbitrary measure space. For example (Theorem 9 of Rao [19]), consider the measure space (ω_1, μ), where countable sets have measure 0, co-countable sets have measure 1, and other sets are not μ-measurable. Then no extension of μ can measure the countable collection of sides of rectangles whose generated σ-algebra contains the well-order on ω_1, since such a measure would contradict Fubini's Theorem. By a related use of Fubini's Theorem, Grzegorek [10] shows that $ME_\omega(X, \mu)$ must fail some atomless measures. Specifically, if κ is the least size of a non Lebesgue measurable subset of the real line, and (X, μ) is any atomless probability space with $|X| = \kappa$, then $ME_\omega(X, \mu)$ is false. The following lemma and corollary also use this method of proof.

Lemma 6.1 *Suppose $Y \subseteq [0,1]$, and suppose $Y^\alpha \nearrow Y$ as $\alpha \nearrow \gamma$, where γ is some limit ordinal and each Y^α is a Lebesgue nullset. Then there is a countable family $\mathcal{E}$ of subsets of $[0,1]$ such that every measure ν extending Lebesgue measure which measures all the sets in $\mathcal{E}$ makes Y a ν-nullset.*

Proof: First, fix $\epsilon > 0$. Let λ be Lebesgue measure on $[0,1]$. For each $\alpha < \gamma$, cover Y^α by an open set, U^α, with $\lambda(U^\alpha) \leq \epsilon$. If $y \in Y$, let V^y be U^α, where α is least such that $y \in Y^\alpha$. If $y \notin Y$, let $V^y = \emptyset$. Now, we have constructed a $V \subseteq [0,1] \times [0,1]$, with every horizontal slice V^y an open set of measure $\leq \epsilon$. Let $\{B_n : n \in \omega\}$ be an open base for $[0,1]$, and let $A_n = A_n(\epsilon) = \{y : B_n \subseteq V^y\}$. Let ν be any extension of λ such that each A_n is ν-measurable. Then V is in the σ-algebra generated by ν-measurable rectangles, and hence is $\nu \times \nu$ measurable, so $\int_0^1 \nu(V_x) d\nu(x) \leq \epsilon$ by Fubini's Theorem. Now, for $x \in Y$, the vertical slice V_x contains all of Y except for a Lebesgue nullset, so $\int_0^1 \nu(V_x) d\nu(x) \geq \nu^*(Y)^2$ (where ν^* is outer measure). Hence, $\nu^*(Y)^2 \leq \epsilon$.

Now, ϵ was arbitrary, so we may let $\mathcal{E} = \{A_n(2^{-i}) : n, i < \omega\}$ ∎

Of course, Lemma 6.1 is trivial unless ME_ω is true.

Corollary 6.2 *ME_ω implies that no Lebesgue measurable set of positive measure is an increasing union of Lebesgue nullsets.*

Next, we make some remarks on the additivities of our measures.

Theorem 6.3 *For any cardinal θ, there is a family $\mathcal{E}$ of θ subsets of θ^+, with the following property: Whenever μ is a probability measure on θ^+ such that each set in $\mathcal{E}$ is μ-measurable and each singleton is a nullset, then θ^+ is a union of θ nullsets.*

Proof. Note that if we allowed $\mathcal{E}$ to have size θ^+, there would be an obvious Ulam [26] matrix argument here, but to get $\mathcal{E}$ of size θ, we need a bit more care.

For each $\alpha < \theta^+$, let R_α well-order θ in type at least α. Let $\mathcal{E}$ be the family of all sets of the form $\{\alpha < \theta^+ : \xi R_\alpha \eta\}$, where $\xi, \eta < \theta$. Fix a measure μ as above. By the standard exhaustion argument, it is sufficient to find a union of θ nullsets which covers some μ-measurable set of positive measure, so we assume that this never happens and derive a contradiction. This argument may be viewed either as an attempt to produce the Ulam matrix by just measuring θ sets, or as an attempt to apply Solovay's [22] Boolean ultrapower technique to the R_α to produce, in some random real extension, a well-order of θ of type θ^+.

Let $\mathcal{I}$ be the family of all $X \subseteq \theta^+$ such that $X \subseteq \bigcup_{\xi<\theta} N_\xi$ for some sequence N_ξ of μ-nullsets. By our assumption on μ, no μ-measurable set of positive measure is in $\mathcal{I}$. Clearly, $\mathcal{I}$ is an ideal, and every union of $\leq \theta$ elements of $\mathcal{I}$ is in $\mathcal{I}$.

Let Σ be the family of all $X \subseteq \theta^+$ such that $X \Delta B \in \mathcal{I}$ for some μ-measurable B. Observe that Σ is a subalgebra of $\mathcal{P}(\theta^+)$, and is closed under $\leq \theta$ unions and intersections. Define a measure ν on Σ so that $\nu(X) = \mu(B)$ for some μ-measurable B with $X \Delta B \in \mathcal{I}$; note that this definition of $\nu(X)$ is independent of the B chosen. Also note that ν is θ^+-additive, in the sense that $\nu(\bigcup_{\xi<\theta} X_\xi) = \sum_{\xi<\theta} \nu(X_\xi)$ whenever the X_ξ are disjoint sets in Σ. In particular, every proper initial segment of θ^+ has measure 0.

For $\alpha < \theta^+$ and $\xi < \theta$, let $E^\alpha_\xi = \{\beta : \alpha < \beta < \theta^+ \ \wedge \ rank(\xi, R_\beta) = \alpha\}$. By induction on $\alpha < \theta^+$, prove that each E^α_ξ is ν-measurable. Now, the E^α_ξ form a ν-measurable Ulam matrix, which yields an immediate contradiction.
∎

In [3], Carlson proves this for the case $\theta \leq \mathfrak{c}$. For this case, his argument is much simpler than ours. He notes that the rows of the Ulam matrix are disjoint, so each row can be countably generated by countably many sets, so the entire $\theta \times \theta^+$ matrix can be countably generated by θ sets.

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