Properties of the Class of Measure Separable Compact Spaces

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Abstract

We investigate properties of the class of compact spaces on which every regular Borel measure is separable. This class will be referred to as MS.

We discuss some closure properties of MS, and show that some simply defined compact spaces, such as compact ordered spaces or compact scattered spaces, are in MS. Most of the basic theory for regular measures is true just in ZFC. On the other hand, the existence of a compact ordered scattered space which carries a non-separable (non-regular) Borel measure is equivalent to the existence of a real-valued measurable cardinal $\leq \mathfrak{c}$.

We show that not being in MS is preserved by all forcing extensions which do not collapse ω_1, while being in MS can be destroyed even by a *ccc* forcing.

§**0. Introduction.** As we learn in a beginning measure theory course, every Borel measure on a compact metric space is separable. It is natural to ask to what extent this generalizes to other compact spaces. It is also true that every Borel measure on a compact metric space is regular. In this paper, we study the class, MS, of compacta, X, with the property that every *regular* measure on X is separable. This contains some simple spaces (such as compact ordered spaces and compact scattered spaces), and has some interesting closure properties. One might also study the class of compacta X such that every *Borel* measure on X is separable, but the theory here is very sensitive to the axioms of set theory;

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for example, the existence of an ordered scattered compactum with a non-separable Borel measure is independent of ZFC (see Theorem 1.1). There is also extensive literature about compacta in which all Borel measures are regular [5].

For the class MS, defined using *regular* measures, there are still some independence results, but most of the basic theory goes through in ZFC.

First, some definitions:

All spaces considered here are Hausdorff.

We shall consider primarily finite Borel measures on compact spaces.

If μ is a Borel measure on X, the measure algebra of (X, μ) is the Boolean algebra of all Borel sets modulo μ-null sets. If μ is finite, then such a measure algebra is also a metric space, with the distance between two sets being the measure of their symmetric difference. Then, we say that μ is *separable* iff this metric space is separable as a topological space.

A Borel measure μ on X is *Radon* iff the measure of compact sets is finite and the measure of each Borel set is the supremum of the measures of its compact subsets. If X is compact, this implies that the measure of each Borel set is also the infimum of the measures of its open supersets. Note that for compact spaces, the Radon measures are simply the regular Borel measures.

The *Baire* sets are the sets in the least σ-algebra containing the open F_σ sets. If X is compact and μ is a finite measure defined on the Baire sets, then μ extends uniquely to a Radon measure (see [8], Theorem 54D), and every Borel set is equal to a Baire set modulo a null set.

Definition. MS is the class of all compact spaces X such that every Radon measure on X is separable.

Observe, by the above remarks, that if X is compact, then X is in MS iff every finite Baire measure on X is separable. We shall primarily be concerned with properties of MS, but we shall occasionally (see Theorem 1.1) remark on finite non-regular Borel measures, in which case non-separability could arise from a large number of non-Baire Borel sets.

If not specified otherwise, we give $[0, 1]$ and $2 = \{0, 1\}$ their usual probability measures, and then $[0, 1]^J$ and 2^J have the usual product measures. These measures defined in the usual way would be defined on the Baire sets, but they then extend to Radon measures. These product measures are in fact *completion regular* – that is, for every Borel set E, there are Baire A, B such that $A \subseteq E \subseteq B$ and $B \backslash A$ is a null set – but we do not need this fact here.

Note that the measure algebras of 2^J and $[0, 1]^J$ are isomorphic whenever J is infinite, and they are separable iff J is countable. So, for uncountable J, 2^J and $[0, 1]^J$ are simple examples of compact spaces which are not in MS.

If μ is a Borel measure on X, and E is a Borel set, then $\mu \restriction E$ is the Borel measure on E defined in the obvious way: $(\mu \restriction E)(B) = \mu(B)$ for Borel $B \subseteq E$. We say that μ is *nowhere separable* iff $\mu \restriction E$ is non-separable for each Borel set E of positive measure.

Our basic notions never assume that non-empty open sets have positive measure, but it is sometimes useful to reduce to this situation. If μ is a Radon measure on the compact space X, let U be the union of all open null sets. By regularity of the measure, U is also a null set, and is hence the largest null set. We call $K = X\backslash U$ the *support* of μ. Note that $\mu(K) = \mu(X)$, and every relatively open non-empty subset of K has positive measure.

The following lemma is sometimes useful to reduce the study of non-separable measures to nowhere separable measures:

Lemma 0.0. If X is compact and μ is a non-separable Radon measure on X, then there is a closed $K \subseteq X$ such that $\mu(K) > 0$, $\mu \restriction K$ is nowhere separable, and every relatively open non-empty subset of K has positive measure.

Proof. By Maharam's Theorem [13], there is a Borel $E \subseteq X$ such that $\mu(E) > 0$ and $\mu \restriction E$ is nowhere separable. We then apply regularity of μ to choose $C \subseteq E$ of positive measure, and let K be the support of $\mu \restriction C$. ★

In §1, we consider some classes of topological spaces which are subclasses of MS, and in §2, we discuss various closure properties of MS.

In §§3,4, we look at the behavior of MS in transitive models of set theory. It is easy to see that the property of *not* being in MS is preserved under any forcing extension which does not collapse ω_1. In §4, we show that being in MS need not even be preserved by *ccc* forcing; assuming the existence of a Suslin tree T, we construct an $X \in MS$ such that forcing with T adds a non-separable Radon measure on X in the generic extension. Of course, since the notion of "compact space" is not absolute for models of set theory, some care must be taken to say precisely what is meant by looking at the same compact space in two different models; this is handled in §3, and in a somewhat different way by Bandlow [1].

We do not know if there is any simple way of expressing "$X \in MS$" without mentioning measures. By the results of §§1,2, there are some simple sufficient conditions for a compact space X to be in MS; for example, it is sufficient that X be a subspace of a countable product of ordered spaces and scattered spaces. By the result of §4, any condition of this form, which is preserved in the passage to a larger model of set theory, cannot be a necessary condition as well (or, at least, cannot be proved to be necessary in ZFC).

§1. Subclasses of MS.

We begin by pointing out some simple sufficient conditions for a compact space to be in MS.

First, recall some definitions. A topological space is *ccc* iff there are no uncountable disjoint families of open subsets of the space. If μ is a Radon measure on a compact space, X, then X need not be ccc, but the support of μ is ccc. A space X is a *LOTS* (linearly ordered topological space) if its topology is the order topology induced by some total order on X.

Theorem 1.0. MS contains every compact X such that X satisfies one of the following.

1. X is second countable (= metric).
2. X is scattered.
3. Every *ccc* subspace of X is second countable.
4. X is Eberlein compact.
5. X is a LOTS.

Proof. Suppose that X is compact and μ is a Radon measure on X.

For (1), fix a countable basis $\mathcal{B}$ for X, which is closed under finite unions, and note that $\mathcal{B}$ is dense in the measure algebra of (X, μ). For (2) and (3), if μ were non-separable, then the K provided by Lemma 0.0 would yield an immediate contradiction. Now, (4) follows because, by Rosenthal [14], every *ccc* Eberlein compact is second countable.

For (5), assume that X is a compact LOTS and that μ is non-separable. By Lemma 0.0 and the fact that every closed subspace of X is a LOTS, we may assume, without loss of generality, that μ is nowhere separable on X; in particular, every point in X is a null set. We may also assume that $\mu(X) = 1$. Let a be the first element of X and b the last element of X. Define $f : X \to [0, 1]$ by: $f(x) = \mu([a, x])$. Then f is continuous (since points are null sets), $f(a) = 0$, and $f(b) = 1$. Let $\lambda = \mu f^{-1}$ be the induced Borel measure on $[0, 1]$. Then λ is regular and separable. Let Σ be the family of all Borel subset B of X such that there is a Borel subset E of $[0, 1]$ with $\mu(B \Delta f^{-1}(E)) = 0$. To conclude that μ is separable (and hence a contradiction), it is sufficient to show that Σ in fact contains all Borel sets, since then the measure algebras of (X, μ) and $([0, 1], \lambda)$ will be isometric. This will follow if we can show that Σ contains all Baire sets. Since Σ is a σ-algebra and every Baire set is in the σ-algebra generated by intervals, it is sufficient to show that Σ contains all intervals. Since Σ certainly contains all singletons (take $E = \emptyset$), it is sufficient to show that each $[a, x] \in \Sigma$. Fix x, and let $s = f(x)$, and $E = f([a, x]) = [0, s]$; then $f^{-1}(E) = [a, z]$ for some $z \geq x$ with $f(z) = s$. $[a, x] \subseteq [a, z]$, and $\mu([a, x]) = f(x) = f(z) = \mu([a, z])$, so $\mu([a, x] \Delta [a, z]) = 0$. ★

The proof of (5) would have been a little nicer if we could have said that f were 1–1, since that would have implied that X is second countable. But we cannot say this. Even if all non-empty open subsets of X have positive measure, there could be points $x < z$ with no points between them, in which case $f(x) = f(z)$. For a specific example, take X to be the double arrow space, which is not second countable but which is the support of a Radon measure.

Regarding (4), the statement that all Corson compacta are in MS is independent of ZFC. See Kunen and van Mill [12] and §2 for further discussion.

The proofs of (2) and (5) involve passing to the support of the measure, by Lemma 0.0, which is justified by regularity of the measure. If we drop regularity, X can be both scattered *and* a LOTS and still have a non-separable Borel measure:

Theorem 1.1. There is a compact scattered LOTS which has a non-separable finite Borel measure iff there is a real-valued measurable cardinal $\leq \mathfrak{c}$.

Proof. If κ is real-valued measurable, let μ be a real-valued measure on κ such that the set of limit ordinals is a null set; then every subset of κ is equal to a Borel (in fact, open) set modulo a null set. This measure is non-separable by the Gitik-Shelah Theorem [4,6,7]. So, μ on $\kappa + 1$ yield an example of an ordered scattered continuum having a non-separable Borel measure.

Now we show that, if there are no real-valued measurable cardinals $\leq \mathfrak{c}$, and μ is a finite Borel measure on a compact scattered LOTS X, then μ is completely atomic.

We do not lose generality if we assume that μ is atomless on X, there are no real-valued measurable cardinals $\leq \mathfrak{c}$, and $\mu(X) = 1$. We derive a contradiction.

Remark: If $S \subseteq X$ has the property that every subset of S is Borel, then $\mu(S) = 0$ (by no real-valued measurable cardinals). More generally, call (S, f, θ) a *dangerous triple* iff S is a Borel subset of X, $\mu(S) > 0$, θ is a cardinal, and $f : S \to \theta$ has the property that $f^{-1}(Z)$ is Borel for each $Z \subseteq \theta$ and $\mu(f^{-1}(\{z\})) = 0$ for all $z \in \theta$. Then the induced measure, μf^{-1}, is a non-trivial measure defined on all subsets of θ, and must then be completely atomic (again, by no real-valued measurable cardinals). This is not immediately a contradiction (unless there are no two-valued measurable cardinals either). But, since μ is atomless, there must be a Borel $Y \subseteq S$ which is not equal to any $f^{-1}(Z)$ (for $Z \subseteq \theta$) modulo a null set. We shall use this remark later.

Let $X^{(\alpha)}$ be the α^{th} derived subset of X. If $x \in X$, let $rank(x)$ be the least α such that $x \notin X^{(\alpha+1)}$. If C is a non-empty closed subset of X, let $rank(C)$ be the least α such that $C \cap X^{(\alpha+1)} = \emptyset$. Note that if $\alpha = rank(C)$, then $C \cap X^{\alpha}$ is finite and non-empty.

Let $\mathcal{C}$ be the set of all closed $C \subset X$ such that $\mu(C) = 0$ and C contains the first and last elements of X. If $C \in \mathcal{C}$, let $\mathcal{I}(C)$ be the set of all non-empty maximal intervals of $X \backslash C$. If $x \in X \backslash C$, let $comp(x, C)$ be the (unique) $I \in \mathcal{I}(C)$ such that $x \in I$. Note that if $C, D \in \mathcal{C}$, $C \subseteq D$, and $x \in X \backslash D$, then $comp(x, D) \subseteq comp(x, C)$.

If $C, D \in \mathcal{C}$, say $C \ll D$ iff $C \subset D$ and for all $x \in X \backslash D$, $rank(cl(comp(x, D))) < rank(cl(comp(x, C)))$ (here, cl denotes topological closure). Observe that if we get $C_n \in \mathcal{C}$ for $n \in \omega$ with each $C_n \ll C_{n+1}$, we will have a contradiction, since $\bigcup_{n \in \omega} C_n$ will have measure 0 and equal X (since an x not in the union would yield a decreasing ω-sequence of ordinals).

Thus, it is sufficient to fix $C \in \mathcal{C}$ and find a $D \in \mathcal{C}$ with $C \ll D$. First, note that if $S \subseteq X \backslash C$ and S contains at most one point from each $I \in \mathcal{I}(C)$, then every subset of S is Borel, so $\mu(S) = 0$. So, $\mu(S) = 0$ whenever S contains at most countably many points from each $I \in \mathcal{I}(C)$.

By expanding C if necessary, we may assume that for each $(a, b) \in \mathcal{I}(C)$, the points of maximal rank in $[a, b]$ are among $\{a, b\}$.

For each $(a, b) \in \mathcal{I}(C)$: If b is a successor point, let $R_0(b)$ be the singleton of its predecessor. If $cf(b) = \omega$, let $R_0(b)$ be some increasing ω-sequence in (a, b) converging to b. Otherwise, let $R_0(b) = \emptyset$. Likewise define $L_0(a)$ to be a singleton if a is a predecessor point, a decreasing ω-sequence if $ci(a) = \omega$, and empty if $ci(a) > \omega$.

Let $\mathcal{F}$ be the set of all closed $D \supseteq C$ such that D is of the form

$$C \cup \bigcup \{R(b) \cup L(a) : (a, b) \in \mathcal{I}(C)\} \quad ,$$

where for each $(a, b) \in \mathcal{I}(C)$: $R(b) = R_0(b)$ if $R_0(b) \neq \emptyset$, and otherwise $R(b)$ is a closed cofinal sequence of type $cf(b)$ in (a, b) converging to b; and, $L(a) = L_0(a)$ if $L_0(a) \neq \emptyset$, and otherwise $L(a)$ is a closed coinitial sequence of type $ci(a)$ in (a, b) converging to a.

Note that $\mathcal{F}$ is closed under countable intersections, so we may fix $D \in \mathcal{F}$ of minimal measure. Then, note that $\mu(D) = 0$. To see this, consider (S, f, θ), where $S = D \backslash C$, $\theta = |\mathcal{I}(C)|$, and f maps $I \cap (D \backslash C)$ to one point in θ for each $I \in \mathcal{I}(C)$. If $\mu(D) > 0$, then (S, f, θ) would be a dangerous triple. But also, note that if $cf(b) > \omega$, then every Borel set either contains or is disjoint from a closed cofinal sequence in b. Using this, and minimality of $\mu(D)$, we see that every Borel $Y \subseteq S$ is equal to some $f^{-1}(Z)$ modulo a null set, which is a contradiction.

So, $C \ll D$. ★

§**2. Closure Properties of** MS**.** In this section, we consider questions about the closure of MS under subspaces, continuous images, continuous pre-images, and products. We begin with:

Lemma 2.0. If $X \in MS$, then every closed subspace of X is in MS.

Of course, this is trivial, since a measure on a subspace induces a measure on X in the obvious way. The same argument works for continuous images, but requires a little care:

Lemma 2.1. Suppose that $X \in MS$ and f is a continuous map from X onto Y. Then Y is in MS.

Proof. Suppose μ were a non-separable Radon measure on Y. Choose a Radon measure ν on X such that $\mu = \nu f^{-1}$. The existence of such a ν follows easily from the Hahn-Banach Theorem plus the Riesz Representation Theorem; see also Henry [10], who proved this, plus some more general results. Now, the measure algebra of μ embeds into the measure algebra of ν, so ν is non-separable, contradicting $X \in MS$. ★

In particular, if X maps onto $[0,1]^{\omega_1}$, then $X \notin MS$. It is a well-known open question of Haydon whether the converse holds; that is, if X is compact and $X \notin MS$, must X map onto $[0,1]^{\omega_1}$? Many counter-examples are known under CH or some other axioms of set theory [2,9,11,12], but it is unknown whether a "yes" answer is consistent, or follows from $MA + \neg CH$.

We shall now show that MS is closed under countable products; it is obviously not closed under uncountable products. First, consider a product of two spaces:

Lemma 2.2. If $X, Y \in MS$, then $X \times Y \in MS$.

Proof. Let λ be a Radon measure on $X \times Y$. We show that λ is separable.

Let μ be the Radon measure on X induced from λ by projection on the first co-ordinate. Since $X \in MS$, there is a countable family $\{D_n : n \in \omega\}$ of closed subsets of X which is dense in the measure algebra of (X, μ).

For each n, let ν_n be the Radon measure on Y induced from $\lambda \restriction (D_n \times Y)$ by projection on the second co-ordinate. Since $Y \in M$, for each n there is a family $\{E^n_m : m \in \omega\}$ of closed subsets in Y which is dense in the measure algebra of (Y, ν_n).

Then the family of the finite unions of the sets of the form $D_n \times E^n_m$ is dense in the measure algebra of $(X \times Y, \lambda)$.★

Theorem 2.3. MS is closed under countable products.

Proof. Suppose that $X_n (n \in \omega)$ are in MS and μ is a Radon measure on $X = \Pi_{n \in \omega} X_n$.

For every n, let π_n denote the natural projection from X onto $Y^n = \Pi_{k \leq n} X_k$. Then $\mu_n = \mu \, \pi_n^{-1}$ is a Radon measure on Y^n, and therefore separable, by the previous Lemma (plus induction). For each n, fix a countable family $\mathcal{D}_n$ which is dense in the measure algebra of (Y^n, μ_n). Then $\mathcal{D} = \bigcup_{n \in \omega} \{\pi_n^{-1}(D) : D \in \mathcal{D}_n\}$ is dense in the measure algebra of (X, μ).★

By the same argument:

Lemma 2.4 MS is closed under inverse limits of countable length.

Since MS is closed under countable products and not closed under uncountable products, it is reasonable to consider now Σ-products, a notion between countable and uncountable products. Let $X_\alpha (\alpha \in \kappa)$ be topological spaces, let X be the usual Tychonov product of the X_α, and let $a = \langle a_\alpha : \alpha \in \kappa \rangle$ be a point in X. We define $\Sigma(a)$ to be the set of all points x of X which differ from a on just a countable set of coordinates. Considered as a subspace of X, this set is called the Σ-*product* of the X_α with *base point* a. If κ is countable, this is just the Tychonov product. If κ is uncountable, then except in trivial cases, $\Sigma(a)$ is not compact and is a proper subset of the Tychonov product. So, the question we address now is: if each $X_\alpha \in MS$, must every compact subspace of $\Sigma(a)$ be in MS? The answer turns out to be independent of ZFC, and in fact equivalent to a weakened version of Martin's Axiom (MA).

Let $MA_{ma}(\omega_1)$ denote the statement that $MA(\omega_1)$ is true for measure algebras; that is, whenever $\mathbb{P}$ is a ccc partial order which happens to be a measure algebra, then one can always find a filter meeting ω_1 dense sets. So, $MA_{ma}(\omega_1)$ implies $\neg CH$, and $MA_{ma}(\omega_1)$ follows from $MA(\omega_1)$. But also, $MA_{ma}(\omega_1)$ is true in the random real model, or in any model with a real-valued measurable cardinal, where most of the combinatorial consequences of full MA fail (see [4]). Consequences of $MA(\omega_1)$ for measure algebras in measure theory are numerous (see [3]), and some of them really only require $MA_{ma}(\omega_1)$.

By Kunen and van Mill [12], $MA_{ma}(\omega_1)$ is equivalent to the fact that all Corson compacta are in MS. Recall that X is called a *Corson compact* iff X is homeomorphic to a compact subspace of a Σ-product of copies of $[0, 1]$. So, if $MA_{ma}(\omega_1)$ fails, there is a compact subspace of a Σ-product of spaces in MS which fails to be in MS. Conversely, we can adapt the proof in [12] to show:

Theorem 2.5. Assuming $MA_{ma}(\omega_1)$, if Y is a compact subspace of a Σ-product of spaces in MS, then $Y \in MS$.

Proof. Suppose that Y is a compact subspace of the Σ-product of the X_α ($\alpha \in \kappa$), with base point a, where each $X_\alpha \in MS$. Assume that μ is a non-separable Radon measure on Y. By Lemma 0.0, we may assume that every non-empty relatively open subset of Y has positive measure. Let $J = \{\alpha \in \kappa : \exists y \in Y (y_\alpha \neq a_\alpha)\}$. If J is countable, then Y is a homeomorphic to a closed subspace of the Tychonov product of the X_α ($\alpha \in J$), so Y would be in MS by Theorem 2.3 and Lemma 2.0. So, we assume J is uncountable and derive a contradiction.

Choose distinct $\alpha_\xi \in J$ for $\xi < \omega_1$. For each ξ, let $\pi_\xi : Y \to X_{\alpha_\xi}$ be the natural projection. For each ξ, there is a $y_\xi \in Y$ with $\pi_\xi(y_\xi) \neq a_{\alpha_\xi}$, and hence there is a relatively open $U_\xi \subseteq Y$ such that $a_{\alpha_\xi} \neq \pi_\xi(\overline{U}_\xi)$.

Since each U_ξ has positive measure, we can apply $MA_{ma}(\omega_1)$ to find an uncountable $L \subseteq J$ such that $\{U_\xi : \xi \in L\}$ has the finite intersection property. L exists because $MA(\omega_1)$ for a ccc partial order implies that the order has ω_1 as a precaliber. Here the order in question is the measure algebra of X.

By compactness, choose $z \in \bigcap_{\xi \in L} \overline{U}_\xi$. Then $z_\xi \neq a_\xi$ for all $\xi \in L$, contradicting the definition of Σ-product. ★

We now consider the situation with continuous preimages of spaces in MS. Suppose X is compact, $f : X \to Y$, and $Y \in MS$. Obviously, we cannot conclude $X \in MS$, since 2^{ω_1} maps onto 2^ω. But we might hope to conclude $X \in MS$ if we know also that the preimage of each point is in MS. Unfortunately, this is false, at least under CH, by an example of Kunen [11]: under CH, there is a closed subset X of 2^{ω_1} such that X supports a non-separable Radon probability measure, yet, the projection $f : X \to 2^\omega$ satisfies that for each $y \in 2^\omega$, $f^{-1}\{y\}$ is second countable.

However, there are two special cases where we can conclude from $f : X \to Y$ that $X \in MS$. One (Theorem 2.7) is where $Y \in MS$ and the point preimages are scattered. The other (Theorem 2.9) is where the point preimages are in MS and Y is scattered. Of course, there is a third special case which we have already covered: if X is a product, $Y \times Z$, and f is projection, it is sufficient that the point preimages (i.e. Z) be in MS to conclude $X \in MS$ by Lemma 2.2.

In the proof of Theorem 2.7, we shall use the following general notation. Suppose X and Y are compact, $f : X \to Y$, and μ is a Radon measure on X. Let $\nu = \mu f^{-1}$ be the induced measure on Y. If E is any Borel subset of X, let ν_E be the measure on Y defined by $\nu_E(B) = \mu(E \cap f^{-1}(B))$. Clearly, $0 \le \nu_E \le \nu$. Let $\delta(E) \in L^1(\nu)$ be the Radon-Nikodym derivative of ν_E; so $d\nu_E = \delta(E)d\nu$. Then $0 \le \delta(E)(x) \le 1$ for ae x. In the following, $\|\cdot\|$ always denotes the L^1 norm on $L^1(\nu)$.

The next lemma shows how to split a closed subset of X into two independent pieces.

Lemma 2.6. Suppose that X and Y are compact and $f : X \to Y$. Suppose that μ is a nowhere separable Radon measure on X, but $\nu = \mu f^{-1}$ is a separable measure on Y. Let $H \subseteq X$ be closed, and fix $\epsilon > 0$. Then there are disjoint closed $K_0, K_1 \subseteq H$ such that for $i = 0, 1$, $\delta(K_i) \le \frac{1}{2}\delta(H)$ and $\|\frac{1}{2}\delta(H) - \delta(K_i)\| \le \epsilon$.

Proof. Let $\mathcal{M}$ be the measure algebra of X, μ. Let $\mathcal{N}$ be the sub σ-algebra of $\mathcal{M}$ generated by H and all $f^{-1}(B)$, where B is a Borel subset of Y. Since $\mathcal{M}$ is nowhere separable while $\mathcal{N}$ is separable, Maharam's Theorem implies that there are complementary Borel sets $E_0, E_1 \subseteq X$ such that $\mu(E_0 \cap A) = \mu(E_1 \cap A) = \frac{1}{2}\mu(A)$ for all $A \in \mathcal{N}$. In particular, whenever $B \subseteq Y$ is Borel, and i is 0 or 1, $\mu(E_i \cap H \cap f^{-1}(B)) = \frac{1}{2}\mu(H \cap f^{-1}(B))$. Thus, $\delta(E_i \cap H) = \frac{1}{2}\delta(H)$.

Now, for $i = 0, 1$, let K_i^n for $n \in \omega$ be an increasing sequence of closed subsets of E_i, such that $\mu(K_i^n) \nearrow \mu(E_i)$. Then $\delta(K_i^n) \to \delta(E_i)$ in $L^1(\nu)$, so, for n sufficiently large, setting $K_i = K_i^n$ will satisfy the Lemma. ★

Theorem 2.7. Suppose that X is compact, $f : X \to Y$, $Y \in MS$, and $f^{-1}\{y\}$ is scattered for all $y \in Y$. Then $X \in MS$.

Proof. Suppose $X \notin MS$. We shall find a $y \in Y$ such that $f^{-1}\{y\}$ is not scattered. Let μ be a non-separable Radon measure on X. We may assume that μ is nowhere separable, since otherwise we may simply replace X by a closed subset of X on which μ is nowhere separable.

We shall find closed subsets of X, H_s, for $s \in 2^{<\omega}$, such that they form a tree:

(1) $H_{()} = X$. For each s, H_{s0} and H_{s1} are disjoint non-empty closed subsets of H_s.

Note, now, that if $y \in \bigcap\{f(H_s) : s \in 2^{<\omega}\}$, then $f^{-1}\{y\}$ has a closed subset which maps onto 2^ω, so $f^{-1}\{y\}$ is not scattered. To ensure that there is such a y, we assume also

(2) For each $n \in \omega$, there is a closed $L_n \subseteq Y$ such that $f(H_s) = L_n$ for all $s \in 2^n$.

Then the L_n will form a decreasing sequence of closed sets, so, by compactness, we may simply choose $y \in \bigcap_{n\in\omega} L_n$. So, we are done if we can actually construct the H_s and L_n to satisfy (1,2). To aid in the inductive construction, we assume also:

(3) $\nu(L_n) > 0$ for all n.
(4) $\delta(H_s) \geq 2^{-2n}$ ae on L_n, for each $s \in 2^n$.

Here, ν and $\delta(H)$ are as defined above. Since items (1-4) are trivial for $n = 0$, we are done if we can show how, given L_n and the H_s for $s \in 2^n$, we can construct L_{n+1} and each H_{s0}, H_{s1}. First, apply Lemma 2.6 and choose, for each s, disjoint closed $K_{s0}, K_{s1} \subseteq H_s$ such that for $i = 0, 1$, $\delta(K_{si}) \leq \frac{1}{2}\delta(H_s)$ and $\|\frac{1}{2}\delta(H_s) - \delta(K_{si})\| \leq 2^{-3n-4}\nu(L_n)$. Let

$$A_{si} = \{y \in L_n : \delta(K_{si})(y) \leq \frac{1}{4} \cdot 2^{-2n}\} \quad .$$

Since $\frac{1}{2}\delta(H_s) \geq \frac{1}{2}2^{-2n}$ on L_n,

$$2^{-3n-4}\nu(L_n) \geq \|\frac{1}{2}\delta(H_s) - \delta(K_{si})\| \geq \nu(A_{si}) \cdot \frac{1}{4}2^{-2n} \quad ,$$

so $\nu(A_{si}) \leq 2^{-n-2}\nu(L_n)$. Let $B = \bigcup\{A_{si} : s \in 2^n, i = 0, 1\}$. Then $\nu(B) \leq \frac{1}{2}\nu(L_n)$, so $\nu(L_n \backslash B) > 0$. For all $y \in L_n \backslash B$, $\delta(K_{si})(y) \geq 2^{-2(n+1)}$ for each s, i. In particular, then, $\nu(L_n \backslash B \backslash f(K_{si})) = 0$. So, we may choose a closed $L_{n+1} \subseteq (L_n \backslash B)$ such that $\nu(L_{n+1}) > 0$ and $L_{n+1} \subseteq f(K_{si})$ for each s, i. Finally, let $H_{si} = K_{si} \cap f^{-1}(L_{n+1})$. Note that $\delta(H_{si}) = \delta(K_{si}) \geq 2^{-2(n+1)}$ on L_{n+1}. ★

Now, before turning to the case that Y is scattered, let us pursue the following idea. If $X \notin MS$, X could still have a clopen subset in MS; for example, X could be the disjoint sum of 2^{ω_1} and 2^{ω}. However, if one deletes all the open subsets of X which are in MS, one gets a "kernel" which is everywhere non-MS by Theorem 2.8.(d) below.

Given a compact X, define

$$ker(X) = X \backslash \bigcup\{U \subseteq X : U \text{ is open and } cl(U) \in MS\} \quad .$$

Theorem 2.8. If X is any compact space:

a. $ker(X)$ is a closed subset of X.
b. If Y is any closed subset of X, then $ker(Y) \subseteq ker(X)$.
c. $X \in MS$ iff $ker(X) = \emptyset$.
d. $ker(ker(X)) = ker(X)$.

Proof. (a) is obvious. (b) follows from Theorem 2.0. For (c), if $ker(X) = \emptyset$, then by compactness, X is a finite union of closed sets in MS, which clearly implies that $X \in MS$.

If (d) fails, fix $p \in ker(X) \backslash ker(ker(X))$. Applying the definition of ker to $ker(X)$, p has a neighborhood U in X such that $cl(U \cap ker(X)) \in MS$; let V be a neighborhood of p in X such that $cl(V) \subseteq U$; then (by Theorem 2.0), $cl(V) \cap ker(X) \in MS$. Since $cl(V) \notin MS$, let μ be a non-separable Radon measure on $cl(V)$. Applying Lemma 0.0, let K be a closed subset of $cl(V)$ such that $\mu(K) > 0$, $\mu \restriction K$ is nowhere separable, and every relatively non-empty relatively open subset of K has positive measure. Then, $K = ker(K)$, and, applying (b), $ker(K) \subseteq ker(X)$, so $K \subseteq cl(V) \cap ker(X)$, contradicting $cl(V) \cap ker(X) \in MS$. ★

Theorem 2.9. Suppose X and Y are compact, Y is scattered, $f : X \to Y$, and the preimages of all points in Y are in MS. Then X is in MS.

Proof. If $X \notin MS$, $ker(X) \neq \emptyset$, so let $y \in f(ker(X))$ be an isolated point in $f(ker(X))$. Then $f^{-1}(y) \cap ker(X)$ is a clopen subset of $ker(X)$, so $f^{-1}(y) \cap ker(X) \notin MS$ by Theorem 2.8(d), so $f^{-1}(y) \notin MS$ by Theorem 2.0. ★

Corollary 2.10. Suppose S is a direct sum of compact spaces X_α, for $\alpha \in \kappa$ (so S is locally compact). Suppose that each $X_\alpha \in MS$. Then any compactification of S with remainder in MS is in MS (in particular, the 1-point compactification).

Proof. Apply Theorem 2.9 with Y being the 1-point compactification of a discrete κ, and f taking each X_α to α and the remainder to the point at infinity. ★

§3. Compact Spaces in Models of Set Theory. In forcing, we frequently discuss the preservation of a property (such as MS) as we pass between two models of set theory. Suppose that $M \subseteq N$ are two transitive models of ZFC, with X a topological space in M. If M thinks that X has some property, we may ask whether N also thinks that the *same space* X has that property. But, since being a space is not a first-order notion, we must be more precise about what "same space" means. There are actually two possible meanings to this, only one of which makes sense in the case of MS.

The first meaning is the most common one, and is frequently used without comment. Formally, a space is a pair, $\langle X, \mathcal{T} \rangle$, where X is a set and $\mathcal{T}$ is a topology on X. If $\langle X, \mathcal{T} \rangle \in M \subseteq N$, and the statement "$\mathcal{T}$ is a topology on X" is true in M, then this statement will not in general be true in N, but it will be true in N that $\mathcal{T}$ is a basis for a topology, $\mathcal{T}'$, on X. In the future, we shall often suppress explicit mention of $\mathcal{T}$ and $\mathcal{T}'$, and simply say something like: "Let X be a space in M, and now consider the same X in N".

However, in dealing with properties of compact spaces, such as MS, it is really the second meaning which is required. If X is a compact space in M (i.e., the statement, "$\langle X, \mathcal{T} \rangle$ is compact" is true relativized to M), then the same X in N is not necessarily compact. For example, if X is $[0,1]^M$ (the unit interval of M), and N has new reals which are not in M, then the same X in N is not compact from the point of view of N; more generally, if N has new reals, then it is only the scattered compact spaces of M which remain compact in N. If X is a compact space in M, we shall define a compact space in N, which we shall call $\Phi_{M,N}(X)$, or just $\Phi(X)$ when M, N are clear from the context. Informally, $\Phi(X)$ will be the compact space in N which "corresponds" to X. In some simple cases, $\Phi(X)$ is the "obvious thing". For example, if X is the unit interval of M, then $\Phi(X)$ is the unit interval of N; if X is the n-sphere in M, then $\Phi(X)$ is the n-sphere in N; if X is the Stone space of a Boolean algebra $B \in M$, then $\Phi(X)$ is the Stone space of the same B as computed within N. But, we must be careful to check that this $\Phi(X)$ is computed for *every* compact X in some natural way. Here, "natural" can be expressed formally in terms of categories. Let CT be the category of compact T_2 spaces and continuous maps. If M is a transitive model of ZFC, CT^M is just the relativized CT, computed within M. Then, $\Phi_{M,N}$ will be a functor from CT^M to CT^N.

$\Phi(X)$ will in fact be computed in N as some compactification of X, so we pause to make some remarks on compactifications. Here, we just work in ZFC, forgetting temporarily about models.

Let $C(X)$ denote the family of all bounded continuous real-valued functions on X. This is a Banach space, and we let $\|f\|$ denote the usual sup norm. Also, $C(X)$ is a Banach algebra under pointwise product. If $\mathcal{S}$ is any non-empty subset of $C(X)$, let $e_{\mathcal{S}}$, or just e, denote the usual evaluation map from X into the cube, $\prod\{[-\|f\|, +\|f\|] : f \in \mathcal{S}\}$; that is, $(e(x))(f) = f(x)$. Let $\beta(X, \mathcal{S})$ be the closure of $e(X)$ in this cube. It is always true that e is continuous. In some cases (for example, if $\mathcal{S}$ separates points and closed sets), e will be a homeomorphic embedding of X, in which case $\beta(X, \mathcal{S})$ is a compactification of X. If $\mathcal{S} = C(X)$ and X is completely regular, then $\beta(X, \mathcal{S}) = \beta(X)$, and we have just given one of the standard definition of the Čech compactification. If X is completely regular, then every compactification of X is of the form $\beta(X, \mathcal{S})$ for some $\mathcal{S}$ – namely, the collection of all those $f \in C(X)$ which extend to the compactification.

If $\mathcal{T} \subseteq \mathcal{S} \subseteq C(X)$, let us use π to denote the natural projection from $\beta(X, \mathcal{S})$ to $\beta(X, \mathcal{T})$. In the case $\mathcal{S} = C(X)$, this is just expressing the maximality of $\beta(X)$ among all compactifications. If $\mathcal{T}$ "generates" $\mathcal{S}$, then π is a homeomorphism. More precisely, let $c(\mathcal{T})$ denote the closure of $\mathcal{T}$ in the Banach algebra $C(X)$; this is the smallest closed linear subspace of $C(X)$ containing $\mathcal{T}$ and closed under pointwise products of functions.

Lemma 3.0. $\beta(X, \mathcal{T})$ and $\beta(X, c(\mathcal{T}))$ are homeomorphic.

Proof. It is easy to check that the projection $\pi : \beta(X, c(\mathcal{T})) \to \beta(X, \mathcal{T})$ is 1-1; that is, if $\psi, \varphi \in \beta(X, c(\mathcal{T}))$ and $\psi(f) = \varphi(f)$ for all $f \in \mathcal{T}$, then $\psi(f) = \varphi(f)$ for all $f \in c(\mathcal{T})$. ★

The functorial properties of these compactifications are a little complicated because of the additional parameter, $\mathcal{S}$. Suppose that X, Y are both compact spaces and $h : X \to Y$ is a continuous function, $\mathcal{S} \subseteq C(X)$, and $\mathcal{T} \subseteq C(Y)$. If we know that $h \circ f \in \mathcal{S}$ for each $f \in \mathcal{T}$, then in a natural way we can define a continuous function $\beta(h, \mathcal{S}, \mathcal{T}) : \beta(X, \mathcal{S}) \to \beta(Y, \mathcal{T})$ by $\beta(h, \mathcal{S}, \mathcal{T})(\psi)(f) = \psi(h \circ f)$.

Returning now to models, let $M \subseteq N$ be two transitive models of ZFC, and we define $\Phi = \Phi_{M,N} : CT^M \to CT^N$ as follows. If $X \in CT^M$, let $\Phi(X)$ be $(\beta(X, C(X) \cap M))^N$. More verbosely, working within N, we have the same space X, and we use $C(X) \cap M$, which is a subset of $C(X)$, to compute a compactification of X, which we are calling $\Phi(X)$. This Φ is functorial in the following sense: Let h be a morphism of CT^M; that is, $h, X, Y \in M$ and, in M, h is a continuous map from X to Y, where $X, Y \in CT^M$. Then in N, $h : X \to Y$ is still continuous, and we may extend it to $\Phi(h) : \Phi(X) \to \Phi(Y)$ by letting $\Phi(h) = \beta(h, C(X) \cap M, C(Y) \cap M)$. It is now easy to check from the definitions that

Lemma 3.1. $\Phi_{M,N}$ is a covariant functor from CT^M to CT^N.

Lemma 3.0 may be used to verify that, as claimed above, $\Phi(X)$ is the "obvious thing". The point is, we often do not need the full $C(X) \cap M$, but may get by with some sub-class.

Lemma 3.2. Suppose that in M, X is compact, $\mathcal{T} \subseteq C(X)$, and $c(\mathcal{T}) = C(X)$. Then in N, $\Phi(X) = \beta(X, \mathcal{T})$.

Proof. Observe that in N, $c(\mathcal{T}) = c(C(X) \cap M)$, and apply Lemma 3.0. ★

We mention two special cases of this. First, suppose in M that X is a compact subset of Euclidean space, $\mathbb{R}^k$. Let $\mathcal{T}$ be the set of the k co-ordinate projections. By the Stone – Weierstrass Theorem (applied in M), $c(\mathcal{T}) = C(X)$. But then in N, $\Phi(X) = \beta(X, \mathcal{T})$, which is just the closure of X computed in the $\mathbb{R}^k$ of N. In particular, if X is, say, the n-sphere of M (so, $k = n + 1$), then $\Phi(X)$ is the n-sphere of N. Second, if X is a compact zero dimensional space in M, we may let B be the clopen algebra of X, so that X is the Stone space of B. In M, let $\mathcal{T}$ be the set of all continuous maps from X into $\{0, 1\}$; then $c(\mathcal{T}) = C(X)$. From this, it is easy to see that in N, $\Phi(X)$ is the Stone space of the same B, computed within N.

It is also easy to see that Φ preserves subspaces and products. Also, if X is a LOTS, then $\Phi(X)$ is the Dedekind completion of the same LOTS; to see this, apply the above method, with $\mathcal{T}$ the set of non-decreasing real-valued functions.

See Bandlow [1] for a somewhat different treatment of Φ.

We turn now to measures. This is easiest to approach via the Riesz Representation Theorem, viewing measures as linear functionals on $C(X)$. If $h \in C(X)^M$, and $X \in CT^M$, then in M, h is a continuous map from X to an interval $[a, b]$. So, in N, we we have $\Phi(h)$, which maps $\Phi(X)$ into $\Phi([a, b])$, which is the $[a, b]$ of N. So, $\Phi(h) \in C(\Phi(X))^N$.

Lemma 3.3. Let X be as above, a compact Hausdorff space in M. In N, Φ is an isometric embedding of $C(X) \cap M$ into $C(\Phi(X))$, and $C(\Phi(X))$ is the closed linear span of $\Phi(C(X) \cap M)$.

In particular, suppose that in M, μ is a Radon measure on the compact space X. Then, via integration, μ defines a positive linear functional on $C(X)$, and by Lemma 3.3, this linear functional extends uniquely to a positive linear functional on the $C(\Phi(X))$ of N, which, by the Riesz Representation Theorem, corresponds uniquely to a Radon measure on $\Phi(X)$. We call this measure $\Phi(\mu)$. Suppose, now, that in M, μ is non-separable. Then, in M we may find, for some fixed $\epsilon > 0$, functions $h_\alpha \in C(X)$ for $\alpha < \omega_1$ such that the $L^1(\mu)$ distance between the h_α is at least ϵ. Then, this same situation will persist in N – that is, in N, $L^1(\Phi(\mu))$ will be non-separable, and hence $\Phi(\mu)$ will be non-separable, assuming that ω_1 has the same meaning in M and N. Thus,

Lemma 3.4. Suppose that $M \subseteq N$ are two transitive models of ZFC, $\omega_1^M = \omega_1^N$, and in M, X is compact and $X \notin MS$. Then in N, $\Phi(X) \notin MS$.

Of course, $\omega_1^M = \omega_1^N$ is necessary. For example, for any X, if the weight of X becomes countable in N, then $\Phi(X)$ will be second countable in N and hence be in MS^N.

The preservation of the property "$X \in MS$" is more tricky, as we discuss in the next section. It is quite possible that $X \in MS^M$, but N is some generic extension of M which adds a new measure which happens to be non-separable. The forcing can even be *ccc*, in which case $\omega_1^M = \omega_1^N$. It is not hard to see that "$X \in MS$" is preserved by any forcing which has ω_1 as a precaliber.

Note that for zero dimensional spaces, the results of this section all reduce to trivialities. If, in M, X is the Stone space of the Boolean algebra B, then $\Phi(X)$ will simply be the Stone space of B as computed in N. Furthermore, if in M, μ is a Radon measure on X, then μ is determined by its values on the clopen sets – i.e., by a finitely additive measure on B – and in N, that same finitely additive measure determines a Radon measure, $\Phi(\mu)$, on $\Phi(X)$.

§**4. Destroying Membership in** MS**.** In this section we show that being in MS can be destroyed by a *ccc* forcing – specifically, by forcing with a Suslin tree. Now, the functor Φ of the previous section is from the CT of the ground model, V, to the CT of a generic extension of V. In the generic extension, X will contain a copy of 2^{ω_1}, which, by Lemma 2.0, will be sufficient to imply that $X \notin MS$.

Theorem 4.0. If there is a Suslin tree, T, then there is a Corson compact space $X \in MS$ such that T forces that X contains a homeomorphic copy of 2^{ω_1}.

Proof. Actually, our proof just uses the fact that T is Aronszajn; except that forcing with an Aronszajn tree does not in general preserve ω_1. In any case, T will force that X contains a homeomorphic copy of 2^λ, where λ is the ω_1 of the ground model, but this is trivial if λ becomes countable in the T extension.

As usual, $Lev_\alpha(T)$ denotes level α of the tree and $T_\alpha = \bigcup_{\xi<\alpha} Lev_\xi(T)$. Let us use $\sqsubseteq$ for the tree order.

We shall construct the space X from the chains of T. Identify $\mathcal{P}(T \times 2)$ with $2^{T\times 2}$ by identifying a set with its characteristic function. Giving $2^{T\times 2}$ its usual topology makes $\mathcal{P}(T\times 2)$ into a compact space. If $x \in \mathcal{P}(T\times 2)$, let $\hat{x} \in \mathcal{P}(T)$ be its projection: $\hat{x} = \{t \in T : \exists i < 2(\langle t, i\rangle) \in x\}$. Let X be the set of all $x \in \mathcal{P}(T \times 2)$ such that $\hat{x}$ is a downward-closed chain in T; that is $\forall s, t \in \hat{x}(s \sqsubseteq t \vee t \sqsubseteq s)$ and $\forall t \in \hat{x} \forall s \sqsubseteq t(s \in \hat{x})$. Note that X is closed, and hence compact. Since T is Aronszajn, each such $\hat{x}$ is countable; so, identifying sets with characteristic functions, every $x \in X$ is eventually 0, so X is a compact subspace of a Σ-product of copies of $\{0, 1\}$, and hence Corson compact (see §2).

Now, it is easy to see that in any extension, $V[G]$, of V, $\Phi(X)$ is just the space defined from the same tree, by the same definition. However, if in $V[G]$, there is an uncountable maximal chain $C \subseteq T$, then $\{x \in \mathcal{P}(T \times 2) : \hat{x} = C\}$ will be a subspace of $\Phi(X)$ homeomorphic to 2^C, which is homeomorphic to 2^{ω_1}.

So, we are done if we can prove (in V) that $X \in MS$. Now, one cannot prove in ZFC that every Corson compact is in MS [12], but this one is.

Let ν be a Radon measure on X. For each $t \in T$, let $X_t = \{x \in X : t \in \hat{x}\}$. This is closed, and hence measurable. If $\epsilon \geq 0$, let $T^\epsilon = \{t \in T : \nu(X_t) > \epsilon\}$. Note that T^ϵ is a sub-tree of T. If $\epsilon > 0$, then each level of T^ϵ is finite (since the X_t are disjoint for t on a given level of T). Since T is Aronszajn, this implies that T^ϵ is countable for each $\epsilon > 0$. Letting $\epsilon \searrow 0$, we see that $T^0 = \{t \in T : \nu(X_t) > 0\}$ is countable.

So, we can fix an $\alpha < \omega_1$ such that for each $s \in Lev_\alpha(T)$, $\nu(X_s) = 0$. Let $F = \bigcup\{X_s : s \in Lev_\alpha(T)\}$; then F is a null set, and $X \backslash F$ is homeomorphic to a subspace of $\mathcal{P}(T_\alpha \times 2)$, and hence second countable. Since every finite Borel measure on a second countable space is separable, ν is separable. ★

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