First Countable Continua and Proper Forcing

Joan E. Hart† and Kenneth Kunen‡§

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Abstract

Assuming the Continuum Hypothesis, there is a compact first countable connected space of weight $\aleph_1$ with no totally disconnected perfect subsets. Each such space, however, may be destroyed by some proper forcing order which does not add reals.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As in [12],

Definition 1.1 A space $X$ is weird iff $X$ is compact and not scattered, and no perfect subset of $X$ is totally disconnected.

A subset $P$ of $X$ is perfect iff $P$ is closed and has no isolated points. As usual, $c$ denotes the (von Neumann) cardinal $2^{\aleph_0}$. Big weird spaces (of size $2^c$) were produced from CH in Fedorchuk, Ivanov, and van Mill [10]. Small weird spaces (of size $\aleph_1$) were constructed from ♦ in [12], which proved:

Theorem 1.2 Assuming ♦, there is a connected weird space which is hereditarily separable and hereditarily Lindelöf.

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†University of Wisconsin, Oshkosh, WI 54901, U.S.A., hartj@uwosh.edu
‡University of Wisconsin, Madison, WI 53706, U.S.A., kuen@math.wisc.edu
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The weird spaces of [12], [10], and the earlier Fedorchuk [9] are all separable spaces of weight $\aleph_1$. Our $\diamondsuit$ example is also first countable, because it is compact and hereditarily Lindelöf. In contrast, the CH weird spaces of [10, 9] have no convergent $\omega$-sequences. We do not know whether CH can replace $\diamondsuit$ in Theorem 1.2, but weakening hereditarily Lindelöf to first countable we do get:

Theorem 1.3 Assuming CH, there is a separable first countable connected weird space of weight $\aleph_1$.

This theorem cannot be proved by a classical CH construction. Classical CH arguments build the item of interest directly from an enumeration in type $\omega_1$ of some natural set of size $c$ (e.g., $\mathbb{R}$, $\mathbb{R}^{<\omega_1}$, etc.). The result, then, is preserved by any forcing which does not add reals. These arguments include any CH proof found in Sierpiński’s text [15], as well as most CH proofs in the current literature, including the constructions of the big weird spaces of [9, 10]. In contrast, every space satisfying Theorem 1.3 is destroyed by some proper forcing order which does not add reals.

Our proof of Theorem 1.3 uses classical CH arguments to make $X$ weird, but then, to make $X$ first countable, we adapt the method of Gregory [11] and Devlin and Shelah [2]. The methods of [11] and [2] are, as Hellsten, Hyttinen, and Shelah [13] pointed out, essentially the same. We review the method in Section 2, and use it to prove Theorem 1.3 in Section 4. Although [11] and [2] derive results from $2^{n_0} < 2^{n_1}$, for Theorem 1.3, we need CH; Section 5 explains why.

In Section 3, we show that each space satisfying Theorem 1.3 can be destroyed by a proper forcing which does not add reals; in $V[G]$, we add a point of uncountable character. More precisely, if $X$ is a compactum in $V$, then in each generic extension $V[G]$, we still have the same set $X$ with the natural topology obtained by using the open sets from $V$ as a base. If $X$ is first countable in $V$, then it must remain first countable in $V[G]$, but $X$ need not be compact in $V[G]$. We get the point of uncountable character in the natural corresponding compact space $\tilde{X}$ in $V[G]$. This compact space determined by $X$ was described by Bandlow [1] (and later in [3, 4, 6]), and can be defined as follows:

Definition 1.4 If $X$ is a compactum in $V$ and $V[G]$ is a forcing extension of $V$, then in $V[G]$ the corresponding compactum $\tilde{X}$ is characterized by:

1. $X$ is dense in $\tilde{X}$.
2. Every $f \in C(X, [0, 1]) \cap V$ extends to an $\tilde{f} \in C(\tilde{X}, [0, 1])$ in $V[G]$.
3. The functions $\tilde{f}$ (for $f \in V$) separate the points of $\tilde{X}$.

In forcing, $\tilde{X}$ denotes the $\tilde{X}$ of $V[G]$, while $X$ denotes the $X$ of $V[G]$.
2 PREDICTORS

For example, if $X$ is the $[0,1]$ of $V$, then $\tilde{X}$ will be the unit interval of $V[G]$; note that in statement (2), asserted in $V[G]$, the “[0,1]” really refers to the unit interval of $V[G]$. If in $V$, we have $X \subseteq [0,1]$, then $\tilde{X}$ is simply the closure of $X$ in the $[0,1]^\kappa$ of $V[G]$. If in $V$, $X$ is the Stone space of a boolean algebra $\mathcal{B}$, then $\tilde{X}$ will be the Stone space, computed in $V[G]$, of the same $\mathcal{B}$. In general, the weights of $X$ and $\tilde{X}$ will be the same (assuming that cardinals are not collapsed), but their characters need not be.

Following Eisworth and Roitman [8, 7], we call a partial order $\mathbb{P}$ totally proper iff $\mathbb{P}$ is proper and forcing with it does not add reals.

**Theorem 1.5** If $X$ is compact, connected, and infinite, and $X$ does not have a Cantor subset, then for some totally proper $\mathbb{P}$: $\mathbb{P} \Vdash \tilde{X}$ is not first countable”.

The proof is in Section 3. Observe the importance of connectivity here. Suppose in $V$ that $X$ is the double arrow space, obtained from $[0,1]$ by doubling the points of $(0,1)$. Then in any $V[G]$, $\tilde{X}$ is the compactum obtained from $[0,1]$ by doubling the points of $(0,1) \cap V$, and is hence first countable.

### 2 Predictors

In the following, $\lambda^{\omega_\alpha}$ denotes the set of functions from $\omega_\alpha$ into $\lambda$. Something like the next definition and theorem is implicit in both of [11, 2]:

**Definition 2.1** Let $\kappa, \lambda$ be any cardinals and $\Psi : \kappa^{<\omega_1} \to \lambda$. If $f \in \kappa^{\omega_1}$, $g \in \lambda^{\omega_1}$, and $C \subseteq \omega_1$, then $\Psi, f$ predict $g$ on $C$ iff $g(\xi) = \Psi(f[\xi])$ for all $\xi \in C$. $\Psi$ is a $(\kappa, \lambda)$–predictor iff for all $g \in \lambda^{\omega_1}$ there is an $f \in \kappa^{\omega_1}$ and a club $C$ such that $\Psi, f$ predict $g$ on $C$.

**Theorem 2.2** The following are equivalent whenever $2 \leq \kappa \leq \mathfrak{c}$ and $2 \leq \lambda \leq \mathfrak{c}$:

1. There is a $(\kappa, \lambda)$–predictor.
2. There is a $(\mathfrak{c}, \mathfrak{c})$–predictor.
3. $2^{\aleph_0} = 2^{\aleph_1}$.

**Proof.** (3) $\rightarrow$ (1): Let $C = \omega_1 \setminus \omega$. List $\lambda^{\omega_1}$ as $\{g_\alpha : \alpha < \mathfrak{c}\}$, and choose $f_\alpha \in \kappa^{\omega_1}$ so that the $f_\alpha|\omega$, for $\alpha < \mathfrak{c}$, are all distinct. Then we can define $\Psi : \kappa^{<\omega_1} \to \lambda$ so that $\Psi(f_\alpha[\xi]) = g_\alpha(\xi)$ for all $\xi \in C$.

(1) $\rightarrow$ (2): Fix a $(\kappa, \lambda)$–predictor $\Psi : \kappa^{<\omega_1} \to \lambda$. We shall define $\Phi : (\kappa^\omega)^{<\omega_1} \to (\lambda^\omega)$ so that it is a $(\kappa^\omega, \lambda^\omega)$–predictor in the sense of Definition 2.1. For $p \in (\kappa^\omega)^\xi$
and \( n \in \omega \), define \( p(n) \in \kappa^\xi \) by: 
\( p(n)(\mu) = (p(\mu))(n) \in \kappa \). Then, for \( p \in (\kappa^\omega)^{<\omega_1} \), define \( \Phi(p) = \langle (\Psi(p(n)) : n \in \omega) \rangle \in \lambda^\omega \).

(2) \( \rightarrow \) (3): Fix a \((\mathfrak{c}, \mathfrak{c})\)-predictor \( \Psi : \mathfrak{c}^{<\omega_1} \rightarrow \mathfrak{c} \). Let \( \Gamma : \mathfrak{c}^{<\omega_1} \times \mathfrak{c}^{<\omega_1} \rightarrow \mathfrak{c} \) be any 1-1 function. If \( K \subseteq \omega_1 \) is unbounded and \( \xi < \omega_1 \), let \( \text{next}(\xi, K) \) be the least element of \( K \) which is greater than \( \xi \).

For each \( B \in \mathfrak{c}^{\omega_1} \), choose \( G(n, B) \), \( F(n, B) \in \mathfrak{c}^{\omega_1} \) and clubs \( C(n, B) \subseteq \omega_1 \) for \( n \in \omega \) as follows: Let \( G(0, B) = B \). Given \( G(n, B) \), let \( C(n, B) \) be club of limit ordinals and let \( F(n, B) \in \mathfrak{c}^{\omega_1} \) be such that \( (G(n, B))(\xi) = \Psi((F(n, B)) \upharpoonright \xi) \) for all \( \xi \in C(n, B) \). Then define \( G(n + 1, B) \) so that

\[
(G(n + 1, B))(\xi) = \Gamma(F(n, B) \upharpoonright \text{next}(\xi, C(n, B)), G(n, B) \upharpoonright \text{next}(\xi, C(n, B)))
\]

for each \( \xi \).

Now, fix \( B, B' \in \mathfrak{c}^{\omega_1} \), and consider the statement:

\[
\forall n \in \omega \left[ (G(n, B))(\xi) = G(n, B')(\xi) \right] \quad (\mathfrak{X}(\xi))
\]

So, \( \mathfrak{X}(0) \) is true trivially, and \( \mathfrak{X}(\xi) \) implies \( \mathfrak{X}(\zeta) \) whenever \( \zeta < \xi \). We shall prove inductively that \( \mathfrak{X}(1) \) implies \( \mathfrak{X}(\eta) \) for all \( \eta < \omega_1 \). If we do this, then \( \mathfrak{X}(1) \) will imply \( B = B' \), so we shall have \( 2^{\aleph_0} = 2^{\aleph_1} \), since there are \( 2^{\aleph_1} \) possible values for \( B \) but only \( 2^{\aleph_0} \) possible values for the sequence \( \langle (G(n, B))(0) : n \in \omega \rangle \).

The induction is trivial at limits, so it is sufficient to fix \( \eta \) with \( 1 \leq \eta < \omega_1 \), assume \( \mathfrak{X}(\eta) \), and prove \( \mathfrak{X}(\eta + 1) \) — that is, \( (G(n, B))(\eta) = (G(n, B'))(\eta) \) for all \( n \). Fix \( n \). For \( \xi < \eta \), we have \( (G(n + 1, B))(\xi) = (G(n + 1, B'))(\xi) \), which implies:

a. \( \text{next}(\xi, C(n, B)) = \text{next}(\xi, C(n, B')) \); call this \( \gamma_\xi \).

b. \( F(n, B) |_{\gamma_\xi} = F(n, B') |_{\gamma_\xi} \).

c. \( G(n, B) |_{\gamma_\xi} = G(n, B') |_{\gamma_\xi} \).

Applying (a) for all \( \xi < \eta \): \( \eta \notin C(n, B) \iff \eta \in C(n, B') \). If \( \eta \notin C(n, B), C(n, B') \), then fix \( \xi \) with with \( \xi < \eta < \gamma_\xi \); now (c) implies \( (G(n, B))(\eta) = (G(n, B'))(\eta) \). If \( \eta \in C(n, B), C(n, B') \), then \( \eta \) is a limit ordinal and (b) implies \( F(n, B)|_{\eta} = F(n, B')|_{\eta} \); now \( (G(n, B))(\eta) = (G(n, B'))(\eta) = \Psi((F(n, B)) \upharpoonright \eta) \).

The non-existence of a \((2, 2)\)-predictor is the weak version of \( \diamondsuit \) discussed by Devlin and Shelah in [2], where they use it to prove that, assuming \( 2^{\aleph_0} < 2^{\aleph_1} \), every ladder system on \( \omega_1 \) has a non-uniformizable coloring. By Shelah [14] (p. 196), each such coloring may be uniformized in some totally proper forcing extension.

A direct proof of (3) \( \rightarrow \) (2), resembling the above proof of (3) \( \rightarrow \) (1), would obtain \( C \) fixed at \( \omega_1 \setminus \{0\} \), since one may choose the \( f_\alpha \) so that the \( f_\alpha(0) \), for \( \alpha < \mathfrak{c} \), are all distinct. Gregory [11] used the failure of (2), with this specific \( C \), to derive a result about trees under \( 2^{\aleph_0} < 2^{\aleph_1} \); see Theorem 3.14 below.
3 Some Totally Proper Orders

We consider forcing posets, $(\mathbb{P}; \leq, 1)$, where $\leq$ is a transitive and reflexive relation on $\mathbb{P}$ and $1$ is a largest element of $\mathbb{P}$. As usual, if $p, q \in \mathbb{P}$, then $p \not\leq q$ means that $p, q$ are compatible (that is, have a common extension), and $p \perp q$ means that $p, q$ are incompatible.

**Definition 3.1** Assume that $X$ is compact, connected, and infinite. Let $\mathbb{K} = \mathbb{K}_X$ be the forcing poset consisting of all closed, connected, infinite subsets of $X$, with $p \leq q$ iff $p \subseteq q$ and $1_\mathbb{K} = X$. In $\mathbb{K}$, define $p \asymp q$ iff $p \cap q = \emptyset$.

Note that $p \perp q$ iff $p \cap q$ is totally disconnected. The stronger relation $p \asymp q$ will be useful in the proof that $\mathbb{K}$ is totally proper whenever $X$ does not have a Cantor subset. First, we verify that $\mathbb{K}$ is separative; this follows easily from the following lemma, which is probably well-known; a proof is in [12]:

**Lemma 3.2** If $P$ is compact, connected, and infinite, and $U \subseteq P$ is a nonempty open set, then there is a closed $R \subseteq U$ such that $R$ is connected and infinite.

In particular, in $\mathbb{K}$, if $p \not\leq q$, then we may apply this lemma with $U = p \setminus q$ to get $r \leq p$ with $r \perp q$, proving the following:

**Corollary 3.3** If $X$ is compact, connected, and infinite, then $\mathbb{K}_X$ is separative and atomless.

We collect some useful properties of the relation $\asymp$ on $\mathbb{K}$ in the following:

**Definition 3.4** A binary relation $\not\asymp$ on a forcing poset is a strong incompatibility relation iff

1. $p \not\asymp q$ implies $p \perp q$.
2. Whenever $p \perp q$, there are $p_1, q_1$ with $p_1 \leq p$, $q_1 \leq q$, and $p_1 \not\asymp q_1$.
3. $p \not\asymp q \& p_1 \leq p \& q_1 \leq q \rightarrow p_1 \not\asymp q_1$.

This definition does not require $\not\asymp$ to be symmetric, but note that the relation $p \not\asymp q \& q \not\asymp p$ is symmetric and is also a strong incompatibility relation.

**Lemma 3.5** The relation $\asymp$ is a strong incompatibility relation on $\mathbb{K}_X$.

**Proof.** Conditions (1) and (3) are obvious. For (2): Suppose that $p \perp q$. Let $F = p \cap q$, which is totally disconnected. Then by Lemma 3.2 there is an infinite connected $p_1 \subseteq p \setminus F$. Likewise, we get $q_1 \subseteq q \setminus F$. \(\blacksquare\)
Definition 3.6 If \( P \) is a forcing poset with a strong incompatibility relation \( \mathcal{E} \), then a strong Cantor tree in \( P \) (with respect to \( \mathcal{E} \)) is a subset \( \{ p_s : s \in 2^{<\omega} \} \subseteq P \) such that each \( p_{s\mu} < p_s \) for \( \mu = 0, 1 \), and each \( p_0 \mathrel{\mathcal{E}} p_{-1} \). Then, \( P \) has the weak Cantor tree property (WCTP) (with respect to \( \mathcal{E} \)) iff whenever \( \{ p_s : s \in 2^{<\omega} \} \subseteq P \) is a strong Cantor tree, there is at least one \( f \in 2^\omega \) such that \( P \) contains some \( q = q_f \) with \( q \leq p_f \rceil_n \) for each \( n \in \omega \).

Note that if \( P \) has the WCTP, then the set of \( f \) for which \( q_f \) is defined must meet every perfect subset of the Cantor set \( 2^\omega \), since otherwise we could find a subtree of the given Cantor tree which contradicts the WCTP.

Lemma 3.7 If \( X \) is compact, connected, and infinite, and \( X \) does not have a Cantor subset, then \( K_X \) has the WCTP.

Definition 3.8 \( P \) has the Cantor tree property (CTP) iff \( P \) has the WCTP with respect to the usual \( \bot \) relation.

\( K_X \) need not have the CTP (see Theorem 5.4). A countably closed \( P \) clearly has the CTP. In the case of trees, the CTP was also discussed in [13] (where it was called “\( \aleph_0 \) fan closed”) and in [12]. The following modifies Lemma 3 of [13] and Lemma 5.5 of [12]:

Lemma 3.9 If \( P \) has the WCTP, then \( P \) is totally proper.

Proof. Define \( q \leq' p \) iff there is no \( r \) such that \( r \leq q \) and \( r \perp p \). When \( P \) is separative, this is equivalent to \( q \leq p \).

Fix a suitably large regular cardinal \( \theta \), and let \( M < H(\theta) \) be countable with \( (P, \leq, 1, \mathcal{E}) \in M \), and fix \( p \in P \cap M \). It suffices (see [8]) to find a \( q \leq p \) such that whenever \( A \subseteq P \) is a maximal antichain and \( A \in M \), there is an \( r \in A \cap M \) with \( q \leq' r \). If \( P \) has an atom \( q \leq p \) such that \( q \in M \), then we are done. Otherwise, then since \( M < H(\theta) \), \( P \) must be atomless below \( p \). Let \( \{ A_n : n \in \omega \} \) list all the maximal antichains which are in \( M \). Build a strong Cantor tree \( \{ p_s : s \in 2^{<\omega} \} \subseteq P \cap M \) such that, \( p_0 \leq p \), and such that, when \( n \in \omega \) and \( s \in 2^n \), \( p_s \) extends some element of \( A_n \cap M \). Then choose \( f \in 2^\omega \) such that there is some \( q \in P \) with \( q \leq p_f \rceil_n \) for each \( n \in \omega \).

Proof of Theorem 1.5. Let \( P = K_X \). Working in \( V[G] \), let \( G' = \{ \tilde{p} : p \in G \} \); then \( \bigcap G' = \{ y \} \) for some \( y \in \tilde{X} \setminus X \). Since \( P \) does not add \( \omega \)-sequences, \( \bigcap E \nsubseteq \{ y \} \) whenever \( E \) is a countable subset of \( G' \). Thus, \( \chi(y, \tilde{X}) \) is uncountable.

These totally proper partial orders yield natural weakenings of PFA:
Definition 3.10 If $\mathcal{P}$ is a class of forcing posets, then $\text{MA}_{\mathcal{P}}(\aleph_1)$ is the statement that whenever $P \in \mathcal{P}$ and $D$ is a family of $\leq \aleph_1$ dense subsets of $P$, then there is a filter on $P$ meeting each $D \in D$.

Trivially, $\text{PFA} \rightarrow \text{MA}_{\text{WCTP}}(\aleph_1) \rightarrow \text{MA}_{\text{CTP}}(\aleph_1)$, but in fact $\text{MA}_{\text{WCTP}}(\aleph_1) \leftrightarrow \text{MA}_{\text{CTP}}(\aleph_1)$ (see Lemma 3.13). Also, $\text{MA}_{\text{CTP}}(\aleph_1) \rightarrow 2^{\aleph_0} = 2^{\aleph_1}$ (see Corollary 3.15), so, the natural iteration of (totally proper) CTP orders with countable supports must introduce reals at limit stages. By the proof of Theorem 5.9 in [12], $\text{PFA}$ does not follow from $\text{MA}_{\text{CTP}}(\aleph_1) + \text{MA}(\aleph_1) + 2^{\aleph_0} = \aleph_2$, which in fact can be obtained by ccc forcing over $L$.

We now consider some CTP trees.

Definition 3.11 Order $\lambda^{<\omega_1}$ by: $p \leq q$ if $p \supseteq q$. Let $\emptyset = \emptyset$, the empty sequence.

So, $\lambda^{<\omega_1}$ is a tree, with the root $\emptyset$ at the top. Viewed as a forcing order, it is equivalent to countable partial functions from $\omega_1$ to $\lambda$. We often view $p \in \lambda^{<\omega_1}$ as a countable sequence and let $lh(p) = \text{dom}(p)$. Then $lh(\emptyset) = 0$.

Kurepa showed that SH is equivalent to the non-existence of Suslin trees. A similar proof shows that $\text{MA}_{\text{CTP}}(\aleph_1)$ is equivalent to the non-existence of Gregory trees:

Definition 3.12 A Gregory tree is a forcing poset $\mathbb{P}$ which is a subtree of $c^{<\omega_1}$ and satisfies:

1. $\mathbb{P}$ has the CTP.
2. $\mathbb{P}$ is atomless.
3. $\mathbb{P}$ has no uncountable chains.

It is easily seen that if any of conditions (1)(2)(3) are dropped, such trees may be constructed in ZFC. However:

Lemma 3.13 The following are equivalent:

1. $\text{MA}_{\text{CTP}}(\aleph_1)$.
2. $\text{MA}_{\text{WCTP}}(\aleph_1)$.
3. There are no Gregory trees.
Proof. (1) → (3): Let $\mathbb{P}$ be a Gregory tree. As with Suslin trees under $\text{MA}(\aleph_1)$, a filter $G$ meeting the sets $D_\xi := \{ p \in \mathbb{P} : \text{lh}(p) \geq \xi \}$ yields an uncountable chain, and hence a contradiction, but to apply $\text{MA}_{\text{CTP}}(\aleph_1)$, we must prove that each $D_\xi$ is dense in $\mathbb{P}$. To do this, induct on $\xi$. The case $\xi = 0$ is trivial. For the successor stages, use the fact that $\mathbb{P}$ is atomless. For the limit stages, use the CTP.

(3) → (2): Fix $\mathbb{P}$ with the WCTP and dense sets $D_\xi \subseteq \mathbb{P}$ for $\xi < \omega_1$. We need to produce a filter $G \subseteq \mathbb{P}$ meeting each $D_\xi$. This is trivial if $\mathbb{P}$ has an atom, so assume that $\mathbb{P}$ is atomless.

Inductively define a subtree $T$ of $2^{<\omega_1}$ together with a function $F : T \to \mathbb{P}$ as follows: $F(\emptyset) = 1_{\mathbb{P}}$. If $t \in T$ and $\text{lh}(t) = \xi$, then $t^0 \in T$ and $t^1 \in T$, and $F(t^0)$, $F(t^1)$ are extensions of $F(t)$ such that each $F(t^0) \notin F(t^1)$; to accomplish this, given $t$ and $F(t)$: first choose two $\bot$ extensions of $F(t)$, then extend these to be $\xi$, and then extend these to be in $D_\xi$. If $\eta < \omega_1$ is a limit ordinal and $\text{lh}(t) = \eta$, then $t \in T$ iff $\forall \xi < \eta \exists t^\xi \in T$ and $\exists q \in \mathbb{P} \forall \xi < \eta [q \leq F(t^\xi)]$; then choose $F(t)$ to be some such $q$.

$T$ is clearly atomless, and $T$ has the CTP because $\mathbb{P}$ has the WCTP. If there are no Gregory trees, then $T$ has an uncountable chain, so fix $g \in 2^{\omega_1}$ such that $g|\xi \in T$ for all $\xi < \omega_1$, and let $G = \{ y \in \mathbb{P} : \exists \xi < \omega_1 [F(g|\xi) \leq y] \}$. ☐

Theorem 3.14 (Gregory [11]) If $2^{\aleph_0} < 2^{\aleph_1}$ then there is a Gregory tree.

Corollary 3.15 $\text{MA}_{\text{CTP}}(\aleph_1)$ implies that $2^{\aleph_0} = 2^{\aleph_1}$.

4 A Weird Space

We now prove Theorem 1.3. The basic construction is an inverse limit in $\omega_1$ steps, and we follow approximately the terminology in [5, 12]. We build a compact space $X_{\omega_1} \subseteq [0, 1]^{\omega_1}$ by constructing inductively $X_\alpha \subseteq [0, 1]^{1+\alpha} \cong [0, 1] \times [0, 1]^\alpha$. Usually, one has $X_\alpha \subseteq [0, 1]^\alpha$ in these constructions, but for finite $\alpha$, the notation will be slightly simpler if we start at stage 0 with $X_0 = [0, 1] = [0, 1]^1$; of course, $1+\alpha = \alpha$ for infinite $\alpha$.

Definition 4.1 $\pi_\alpha^\beta : [0, 1]^{1+\beta} \to [0, 1]^{1+\alpha}$ is the natural projection.

As usual, $\pi : X \to Y$ means that $\pi$ is a continuous map from $X$ onto $Y$. These constructions always have $\pi_\alpha^\beta(X_\beta) = X_\alpha$ whenever $0 \leq \alpha \leq \beta \leq \omega_1$. This determines $X_\gamma$ for limit $\gamma$, so the meat of the construction involves describing how to build $X_{\alpha+1}$ given $X_\alpha$. 
A classical CH argument can ensure that $X_{\omega_1}$ is weird, but by Theorem 1.5, such an argument cannot make $X_{\omega_1}$ first countable. However, the same classical argument will let us construct a binary tree of spaces, resulting in a weird space $X_g \subseteq [0,1]^{\omega_1}$ for each $g \in 2^{\omega_1}$. We shall show that if no $X_g$ were first countable, then there would be a $(\gamma,2)$-predictor $\Psi : [0,1]^{<\omega_1} \rightarrow 2$; so CH ensures that some $X_g$ is first countable.

Our tree will give us an $X_p$ for each $p \in 2^{<\omega_1}$. We now list requirements (R1)(R2)(R3)· · ·(R17) on the construction; a proof that all the requirements can be satisfied, and that they yield a weird space, concludes this section. We begin with the requirements involving the inverse limit:

R1. $X_\emptyset = [0,1]$, where $\emptyset$ is the empty sequence.
R2. $X_p$ is an infinite closed connected subspace of $[0,1]^{1+\text{lh}(p)}$.
R3. $\pi_\beta | X_p : X_p \rightarrow X_p |_\alpha$, and is irreducible, whenever $\beta = \text{lh}(p) \geq \alpha$.

When $\gamma = \text{lh}(p) \leq \omega_1$ is a limit, (R2)(R3) force:

$$X_p = \{x \in [0,1]^\gamma : \forall \alpha < \gamma [\pi_\alpha(x) \in X_p |_\alpha]\}.$$  \hfill (1)

To simplify notation for the restricted projection maps, we shall use:

**Definition 4.2** If $\beta = \text{lh}(p) \geq \alpha$ and $r = p | \alpha$, define $\pi_r^p = \pi_\beta^| X_p : X_p \rightarrow X_r$.

As in [12], each of $X_{p^{-0}}$ and $X_{p^{-1}}$ is obtained from $X_p$ as the graph of a “$\sin(1/x)$” curve. We choose $h_q, u_q$, and $v^q_n$ for $n < \omega$ and $q \in 2^{<\omega_1}$ of successor length, satisfying, for $i = 0, 1$:

R4. $u_{p^{-i}} \in X_p$ and $h_{p^{-i}} \in C(X_p \setminus \{u_{p^{-i}}\}, [0,1])$ and $X_{p^{-i}} = \overline{\{h_{p^{-i}}\}}$.
R5. $v^q_{p^{-i}} \in X_p \setminus \{u_{p^{-i}}\}$, and $\langle v^q_n : n \in \omega \rangle \rightarrow u_{p^{-i}}$, and all points of $[0,1]$ are limit points of $\langle h_{p^{-i}}(v^q_n) : n \in \omega \rangle$.

As usual, we identify $h_{p^{-i}}$ with its graph. So, if $\alpha = \text{lh}(p)$, then $X_{p^{-i}}$ is a subset of $[0,1]^{1+\alpha} \times [0,1]$, which we identify with $[0,1]^{1+\alpha+1}$. We shall say that the point $u_{p^{-i}}$ gets expanded in the passage from $X_p$ to $X_{p^{-i}}$; the other points get fixed. (R3) follows from (R4) plus (1). Also, if $\delta < \alpha$, then $\pi^p_{\delta} : X_p \rightarrow X_p |_{\delta}$, and $(\pi^p_{\delta})^{-1}\{x\}$ is a singleton unless $x$ is in the countable set $\{\pi^p_{\delta|\xi}(u_{p|\xi(\xi+1)}) : \delta \leq \xi < \alpha\}$.

We now explain how points in $X_g \subset [0,1]^{\omega_1}$ can predict $g$, in the sense of Definition 2.1. We shall get $A_q$ and $B_q$ for $q \in 2^{<\omega_1}$ of successor length, satisfying:
R6. For $i = 0, 1$: $A_{p^{-i}} \subseteq X_p$ and $A_{p^{-i}} = X_p \setminus B_{p^{-i}}$.

R7. For $i = 0, 1$ and $\xi < \text{lh}(p)$: $A_{p^{-i}} \supseteq (\pi^p_p(\xi))^{-1}(A_{p^{1}((\xi + 1))})$.

R8. $B_{p^{-0}} \cap B_{p^{-1}} = \emptyset$

R9. For $i = 0, 1$: $u_{p^{-i}} \in B_{p^{-i}}$.

Observe that some care must be exercised here in the inductive construction; otherwise, at some stage (R7) might imply that $A_{p^{-i}} = X_p$, so that $B_{p^{-i}} = \emptyset$, making (R9) impossible.

(R6)(R7)(R9) imply that points in $A_{p^{-i}}$ are forever fixed in the passage from $X_p$ to any future $X_q$ with $q \leq p^{-i}$; only points in $B_{p^{-i}}$ can get expanded. Points which are forever fixed must wind up having countable character, and (R8) lets us use a point of uncountable character in $X_g$ to predict $g$:

**Lemma 4.3** Assume that we have (R1 – R9), and assume that $2^{\aleph_0} < 2^{\omega_1}$. Then $X_g$ is first countable for some $g \in 2^{\omega_1}$.

**Proof.** We shall define $\Psi : [0, 1]^{<\omega_1} \rightarrow 2$, and prove that $\Psi$ is a $(c, 2)$-predictor if every $X_p$ contains a point of uncountable character.

Say $\text{lh}(p) = \alpha < \omega_1$ and $\delta < \alpha$. If $x \in B_{p^{-i}} \subseteq X_p$ then, by (R6)(R7), $\pi^g_p(x) \in B_{p^{1}((\delta + 1))} \subseteq X_p|\delta$. Applying (R8), if $x \in [0, 1]^{1+\alpha}$ and $x \in B_{p^{-i}} \cap B_{r^{-j}}$, then $p = r$ and $i = j$; to prove this, consider the least $\delta < \alpha$ such that $p(\delta) \neq r(\delta)$.

Set $\Psi(x) = 0$ if $\text{lh}(x) < \omega$. Now, say $x \in [0, 1]^\alpha$, where $\omega \leq \alpha < \omega_1$ (so $1 + \alpha = \alpha$). If there exist $p \in 2^\alpha$ and $i \in 2$ such that $x \in B_{p^{-i}}$, then these $p, i$ are unique, and set $\Psi(x) = i$. If there are no such $p, i$, then set $\Psi(x) = 0$.

Now, assume that for each $g$, we can find $z = z_g \in X_g$ with $\chi(z, X_g) = \aleph_1$. Let $C = \omega_1 \setminus \omega$. We shall show that $\Psi, z$ predict $g$ on $C$. For $\xi \in C$, let $p^{-i} = g|((\xi + 1))$. Then $z|\xi = \pi^g_p(z) \in X_p$, and $z|\xi$ must be in $B_{p^{-i}}$, since if it were in $A_{p^{-i}}$, then $(\pi^g_p)^{-1}(\pi^g_p(z)) = \{z\}$, so that $\chi(z, X_g) = \aleph_0$. Thus, $\Psi(z|\xi) = i = g(\xi)$. \(\blacksquare\)

Since every $X_g$ clearly has weight $\aleph_1$, we are done if we can make every $X_g$ weird. Since points in $A_{p^{-i}}$ are forever fixed, we must make sure that $A_{p^{-i}}$ has no Cantor subsets. Conditions (R6)(R8) say that $A_{p^{-0}} \cup A_{p^{-1}} \subseteq X_p$, so $A_{p^{-0}}$ and $A_{p^{-1}}$ must be Bernstein sets. Note that Condition (R7) may present a problem at limit stages. When $\text{lh}(p) = \alpha$ we have $A_{p^{-i}} \supseteq \bigcup_{\xi < \alpha} (\pi^p_p(\xi))^{-1}(A_{p^{1}((\xi + 1))})$. Points in $A_{p^{1}((\xi + 1))}$ are forever fixed, so each $(\pi^p_p(\xi))^{-1}(A_{p^{1}((\xi + 1))})$ will have no Cantor subsets. Without further requirements, though, $\bigcup_{\xi < \alpha} (\pi^p_p(\xi))^{-1}(A_{p^{1}((\xi + 1))})$ may contain a Cantor subset. So, we make sure each such union is disjoint from some set in a tree of Bernstein sets:
Definition 4.4 For any topological space \(Y\) and \(p \in 2^{<\omega_1}\), a Bernstein tree in \(Y\) rooted in \(p\) is a family of subsets of \(Y\), \(\{D^q : q \leq p\}\), satisfying:

1. For each \(q\), neither \(D^q\) nor \(Y \setminus D^q\) contains a Cantor subset.
2. Each \(D^q \cap D^q = \emptyset\).
3. If \(r \leq q\) then \(D^r \subseteq D^q\).

Note that if \(Y\) itself does not contain a Cantor subset, then (1) is trivial, and we may take all \(D^q = \emptyset\) to satisfy (2) and (3).

Now, in our construction, we also build \(D^q_p\) for \(q \leq p \in 2^{<\omega_1}\) satisfying:

R10. For each \(p \in 2^{<\omega_1}\): \(\{D^q_p : q \leq p\}\) is a Bernstein tree in \(X_p\) rooted in \(p\).
R11. If \(q \leq p \leq r\) and \(\pi = \pi_r^p : X_p \twoheadrightarrow X_r\) and \(x \in X_p\) with \(\pi^{-1}(\pi(x)) = \{x\}\), then \(x \in D^q_p\) iff \(\pi(x) \in D^q_r\).
R12. For each \(p \in 2^{<\omega_1}\) and \(i \in 2\): \(B^p_{p^{-i}} = D^p_{p^{-i}}\) and \(A^p_{p^{-i}} = X_p \setminus D^p_{p^{-i}}\).

Of course, (R12) simply defines \(A^p_{p^{-i}}\) in terms of the \(D^q_p\), and then (R10) guarantees that no \(A^p_{p^{-i}}\) has a Cantor subset, but we need to verify that the conditions (R1 – R12) can indeed be satisfied. First, three easy lemmas about Bernstein trees. A standard inductive construction in \(\mathfrak{c}\) steps shows:

Lemma 4.5 If \(Y\) is a separable metric space, then there is a Bernstein tree in \(Y\) rooted in \(1\).

Using the fact that every uncountable Borel subset of the Cantor set contains a perfect subset, we get:

Lemma 4.6 Assume that \(Y\) is any topological space, \(Z\) is a Borel subset of \(Y\), and \(\{D^q : q \leq p\}\) is a family of subsets of \(Y\) satisfying (2)(3) of Definition 4.4. Then \(\{D^q : q \leq p\}\) is a Bernstein tree in \(Y\) iff both \(\{D^q \cap Z : q \leq p\}\) is a Bernstein tree in \(Z\) and \(\{D^q \setminus Z : q \leq p\}\) is a Bernstein tree in \(Y \setminus Z\).

Combining these two lemmas:

Lemma 4.7 If \(Y\) is a separable metric space, \(Z\) is a Borel subset of \(Y\), and \(\{E^q : q \leq p\}\) is a Bernstein tree in \(Z\) rooted in \(p\), then there is a Bernstein tree \(\{D^q : q \leq p\}\) in \(Y\) rooted in \(p\) such that each \(D^q \cap Z = E^q\).

Returning to the construction:
Lemma 4.8 There exist $X_p$ for $p \in 2^{<\omega_1}$ satisfying Conditions (R1 – R12).

Proof. We start with $X_\emptyset = [0, 1]$, and we obtain the $D_p^q$ by applying Lemma 4.5.

If $p = \text{lh}(p) > 0$ and we have done the construction for $p|\xi$ for all $\xi < \text{lh}(p)$, then $X_p$ is determined either by (R4) when $\text{lh}(p)$ is a successor or by (\dagger) when $\text{lh}(p)$ is a limit. If $\alpha < \omega_1$, we construct the $D_p^q$ to satisfy (R10)(R11) as follows:

For $\xi < \alpha$, use $\pi_\xi$ for $\pi_p^{\flat}\xi$. Let $Z_\xi = \{ x \in X_p : \pi_\xi^{-1}(\pi_\xi(x)) = \{ x \} \}$, and let $Z = \bigcup_{\xi < \alpha} Z_\xi$. Observe that $Z$ and all the $Z_\xi$ are Borel sets. Let $\{ E_p^q : q \leq p \}$ be the Bernstein tree in $Z$ rooted in $p$ defined by saying that for $x \in Z_\xi$: $x \in E_p^q$ iff $\pi_\xi(x) \in D_p^q$. Note that, by (R11) applied inductively, this is independent of which $\xi$ is used. To obtain the $D_p^q$ from the $E_p^q$, apply Lemma 4.7. Note that, by (R11) applied inductively once again, these $D_p^q$ work for $X_p$.

The $A_{p-i}$ and $B_{p-i}$ (for $i = 0, 1$) are now defined by (R12), and we must verify that this definition satisfies (R7): Assume that $\xi < \text{lh}(p) = \alpha$ and $x \in X_p$ and $\pi_\xi(x) \in A_{p|\xi+1}$. We must show that $x \in A_{p-i}$; equivalently, by (R12), that $x \notin D_p^{p-i}$. Now $\pi_\xi(x) \in A_{p|\xi+1}$ implies that $\pi_\xi^{-1}(\pi_\xi(x)) = \{ x \}$ (using (R4 – R9) inductively), so that $x \notin D_p^{p-i}$ iff $\pi_\xi(x) \notin D_p^{p-i}_q$. By (R10) for $p|\xi$ and Definition 4.4(3), $D_p^{p-i}_q \subseteq D_p^{p|\xi+1}$. So $A_{p|\xi+1} = X_p|\xi \setminus D_p^{p|\xi+1}$ gives us (R7).

Since the $B_{p-i}$ are nonempty, there is no problem choosing the $u_{p-i}$, $v_{p-i}$, and $h_{p-i}$ to satisfy (R4)(R5)(R9), and then the $X_{p-i}$ are defined by (R4).

Finally, we must make each $X_g$ weird. Observe:

Lemma 4.9 Conditions (R1 – R5) imply that if $F \subseteq X_p$ is closed and connected then $(\pi_p^q)^{-1}(F)$ is connected for all $q \leq p$.

Now, we shall make sure that whenever $F$ is a perfect subset of $X_g$, there is some $\alpha < \omega_1$ such that $(\pi_{g|\alpha+1})^{-1}(\{ u_{g|\alpha+1} \} \times [0, 1]) \subseteq F$ (recall that our construction gave us $\{ u_{g|\alpha+1} \} \times [0, 1] \subseteq X_{g|\alpha+1} \subseteq X_{g|\alpha} \times [0, 1]$). By Lemma 4.9, this implies that $F$ is not totally disconnected. The argument in [12] obtained this $\alpha$ by using $\diamond$ to capture $F$. Here, we replace this use of $\diamond$ by a classical CH argument. First, as in [12], construct $\mathcal{F}_p$ for $p \in 2^{<\omega_1}$ so that:

R13. $\mathcal{F}_p$ is a countable family of uncountable closed subsets of $X_p$.

R14. If $F \in \mathcal{F}_p$ and $q \leq p$ then $(\pi_p^q)^{-1}(F) \in \mathcal{F}_q$.

R15. For each $F \in \mathcal{F}_p$, either $u_{p-i} \notin F$, or $u_{p-i} \in F$ and $v_{p-i}^{n} \in F$ for all but finitely many $n$.

R16. $\{ u_{p-i} \} \times [0, 1] \in \mathcal{F}_{p-i}$.
4 A WEIRD SPACE

We may satisfy (R13)(R14)(R16) simply by defining
\[ \mathcal{F}_p = \left\{ (\pi_p^{\xi})^{-1}(u_{p(\xi+1)}) : \xi < \text{lh}(p) \right\}. \]

Requirements (R4)(R14)(R15) imply:

**Lemma 4.10** \( \pi_p^q : (\pi_p^q)^{-1}(F) \to F \) is irreducible whenever \( F \in \mathcal{F}_p \) and \( q \leq p \).

Then, we use CH rather than \( \diamond \) to get:

R17. Whenever \( p \in 2^{<\omega_1} \) and \( F \) is an uncountable closed subset of \( X_p \), there is a \( \beta \) with \( \text{lh}(p) < \beta < \omega_1 \) such that for all \( q < p \) with \( \text{lh}(q) = \beta \) and for each \( x \in \{u_{q,0}, u_{q,-1}\} \cup \{v_{q,i}^n : n \in \omega \land i \in 2\} \), the projections \( \pi = \pi_p^q \) satisfy \( \pi(x) \in F \) and \( |\pi^{-1}(\pi(x))| \leq 1 \).

**Proof of Theorem 1.3.** Assuming that we can obtain (R1 – R17), note that each \( X_g \) is separable, because each \( \pi_p^q : X_g \to X_3 \) is irreducible. Then, to finish, by Lemma 4.3, it suffices to show that each \( X_g \) is weird. Fix a perfect \( H \subseteq X_g \); we shall show that it is not totally disconnected. First, fix \( \alpha < \omega_1 \) such that, if we set \( p = g|\alpha \) and \( F = \pi_p^q(H) \), then \( F \) is perfect (the set of all such \( \alpha \) form a club). Then, fix \( \beta > \alpha \) as in (R17), let \( q = g|\beta \), and let \( i = g(\beta) \), so that \( q^{-i} = g \upharpoonright (\beta + 1) \). Let \( K = (\pi_p^q)^{-1}(F) \). Then \( \pi_p^q(H) \subseteq K \), and this inclusion may well be proper. However, \( u_{q,-i} \in \pi_p^q(H) \) and \( v_{q,-i}^n \in \pi_p^q(H) \) for each \( n \in \omega \) because \( \pi(u_{q,-i}) \in F \) and \( \pi(v_{q,-i}^n) \in F \) and \( |\pi^{-1}(\pi(u_{q,-i}))| = |\pi^{-1}(\pi(v_{q,-i}^n))| = 1 \). It follows (using (R5)) that \( E := \{u_{q,-i}\} \times [0,1] \subseteq \pi_p^q(H) \). Since \( E \in \mathcal{F}_{q^{-i}} \) by (R16) and \( H \) maps onto \( E \), Lemma 4.10 implies that \( (\pi_p^{q^{-i}})^{-1}(E) \subseteq H \). Since \( (\pi_p^{q^{-i}})^{-1}(E) \) is connected by Lemma 4.9, \( H \) cannot be totally disconnected.

Next, to obtain conditions (R1 – R17), we must augment the proof of Lemma 4.8: Fix in advance a map \( \psi \) from \( \omega_1 \setminus \{0\} \) onto \( \omega_1 \times \omega_1 \), such that \( \alpha < \beta \) whenever \( \psi(\beta) = (\alpha, \xi) \). Now, given \( X_p \), use CH and let \( \{F_p^\xi : \xi < \omega_1\} \) be a listing of all uncountable closed subsets of \( X_p \). Whenever \( 0 < \beta < \omega_1 \) and \( \psi(\beta) = (\alpha, \xi) \) and \( \text{lh}(q) = \beta \), we may set \( p = q|\alpha \) and \( F = F_p^\xi \subseteq X_p \). It is sufficient to show how to accomplish (R17) with these specific \( \alpha, \beta, p, q, F \).

Choose a perfect \( K \subset F \) which is disjoint from \( \{\pi_p^{q(\xi)}(u_{q(\xi+1)}) : \alpha \leq \xi < \beta\} \). Then \( \pi_p^q \) is 1-1 on \( (\pi_p^q)^{-1}(K) \), so choosing all \( u_{q,-i} \) and \( v_{q,-i}^n \) in \( (\pi_p^q)^{-1}(K) \) will ensure (R17). Now fix \( i \in 2 \), and write \( u \) and \( v^n \) for \( u_{q,-i} \) and \( v_{q,-i}^n \). To ensure (R15) and (R9), we modify the argument of [12]. Let \( \{Q_n : n \in \omega\} \) list \( \mathcal{F}_q \). Let \( d \) be a metric on \( (\pi_p^q)^{-1}(K) \). For each \( s \in 2^{<\omega} \), choose a perfect \( L_s \subseteq (\pi_p^q)^{-1}(K) \). Make these into a tree, in the sense that each \( L_{s-0} \cap L_{s-1} = \emptyset \), each \( \text{diam}(L_s) \leq 2^{-\text{lh}(s)} \), and
5 Remarks and Examples

One cannot replace “CH” by “$2^{\aleph_0} < 2^{\aleph_1}$” in the statement of Theorem 1.3, since by Proposition 5.3, it is consistent with any cardinal arithmetic that every non-scattered compactum of weight less than $\mathfrak{c}$ contains a copy of the Cantor set. As usual, define

Definition 5.1 $\text{cov}(\mathcal{M})$ is the least $\kappa$ such that $\mathbb{R}$ is the union of $\kappa$ meager sets.

Note that $\text{cov}(\mathcal{M})$ is the least $\kappa$ such that MA($\kappa$) for countable partial orders fails. Using this, we easily see:

Lemma 5.2 If $\kappa < \text{cov}(\mathcal{M})$ and $E_\alpha \subset [0,1]$ is meager for each $\alpha < \kappa$, then $[0,1] \setminus \bigcup_{\alpha < \kappa} E_\alpha$ contains a copy of the Cantor set.

Proposition 5.3 If $X$ is compact and not scattered, and $w(X) < \text{cov}(\mathcal{M})$, then $X$ contains a copy of the Cantor set.

Proof. Replacing $X$ by a subspace, we may assume that we have an irreducible map $\pi : X \to [0,1]$. Let $\mathcal{B}$ be an open base for $X$ with $|\mathcal{B}| < \text{cov}(\mathcal{M})$ and $\emptyset \neq \mathcal{B}$. Whenever $U, V \in \mathcal{B}$ with $U \cap V = \emptyset$, let $E_{U,V} = \pi(U) \cap \pi(V)$. Then $E_{U,V} \subset [0,1]$ is nowhere dense because $\pi$ is irreducible. Applying Lemma 5.2, let $K \subset [0,1]$ be a copy of the Cantor set disjoint from all the $E_{U,V}$. Note that $|\pi^{-1}(y)| = 1$ for all $y \in K$. Thus, $\pi^{-1}(K)$ is homeomorphic to $K$.

5 Remarks and Examples

If $H \subseteq X_g$ is closed and for some initial segment $p = g|\alpha$ the projection $\pi^g_p(H) \in \mathcal{F}_p$, then, by irreducibility, $H = (\pi^g_p)^{-1}(\pi^g_p(H))$, so that $H$ is a $G_\delta$. To make $X_g$ hereditarily Lindelöf, it suffices to capture projections for each closed $H \subseteq X_g$ this way, but it is not clear whether this can be done without using ♦.

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Note that one can force “$\text{cov}(\mathcal{M}) = \mathfrak{c}$” by adding $\mathfrak{c}$ Cohen reals, which does not change cardinal arithmetic, but in the statement of Proposition 5.3, “$\text{cov}(\mathcal{M})$” cannot be replaced by “$\mathfrak{c}$”. If CH holds in $V$, then one may force $\mathfrak{c}$ to be arbitrarily large by adding random reals, and any random real extension $V[G]$ will have
a compact non-scattered space of weight $\aleph_1$ which does not contain a Cantor subset. In fact, Dow and Fremlin [4] show that if $X$ is a compact F-space in $V$, then in a random real extension $V[G]$, the corresponding compact space $\tilde{X}$ has no convergent $\omega$-sequences, and hence no Cantor subsets.

The weird space constructed in [12] also failed to satisfy the CSWP (the complex version of the Stone–Weierstrass Theorem). Using the method there, we can modify the proof of Theorem 1.3 to get:

**Theorem 5.4** Assuming CH, there is a separable first countable connected weird space $X$ of weight $\aleph_1$ such that $X$ fails the CSWP and $\mathbb{K}_X$ fails the CTP.

**Proof.** First, in the proof of Theorem 1.3, replace $[0,1]$ by $\overline{D}$, the closed unit disc in the complex plane, so that we may view $X$ as a subspace of the $\aleph_1$-dimensional polydisc. Then, as in [12], by carefully choosing the functions $h_{p^{-i}}$, one can ensure that the restriction to $X$ of the natural analog of the disc algebra refutes the CSWP of $X$. To refute the CTP, construct in $\overline{D}$ a Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{K}_{\overline{D}}$ such that each $p_s$ is a wedge of the disc with center 0 and radius $2^{-\text{lh}(s)}$; then each $\bigcap_{n \in \omega} p_{f/n} = \{0\}$. Then, since we may assume the point 0 is not expanded in the construction of $X$, the inverse images of the $p_s$ yield a counterexample to the CTP of $\mathbb{K}_X$.

**References**


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