Transfinite Sequences of Continuous and Baire Class 1 Functions

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March 11, 2002

Abstract

The set of continuous or Baire class 1 functions defined on a metric space $X$ is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space $X$.

Introduction

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set $X$ is partially ordered by the pointwise order; that is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Then, $f < g$ iff $f \leq g$ and $g \not\leq f$; equivalently, $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in $\mathcal{F}$ with respect to this order. A classical theorem (see Kuratowski [7], §24.III, Theorem 2) asserts that if $\mathcal{F}$ is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space $X$ (that is, a complete separable metric space), then there exists a monotone sequence of length $\xi$ in $\mathcal{F}$ iff $\xi < \omega_1$. P. Komjáth [5] proved that the corresponding question concerning Baire class $\alpha$ functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

*Partially supported by NSF Grant DMS-0097881.
2000 Mathematics Subject Classification: Primary 26A21; Secondary 03E17, 54C30.
Key words and Phrases: Baire class 1, separable metric space, transfinite sequence of functions.
In the present paper we investigate what happens if we replace the Polish space $X$ by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space $X$, there exists a chain in $C(X, \mathbb{R})$ of order type $\xi$ iff $|\xi| \leq d(X)$. Here, $|A|$ denotes the cardinality of the set $A$, while $d(X)$ denotes the density of the space $X$, that is

$$d(X) = \max \{ \min \{|D| : D \subseteq X \text{ and } \overline{D} = X \} , \omega \} .$$

In particular, for separable $X$, every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some separable metric spaces, there are well-ordered chains of every order type less than $\omega_2$. Furthermore, the existence of chains of type $\omega_2$ and longer is independent of $ZFC + \neg CH$. Under $MA$, there are chains of all types less than $\mathfrak{c}^+$, whereas in the Cohen model, all chains have type less than $\omega_2$.

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set $\mathcal{F}$. This problem was posed by M. Laczkovich, and is considered in detail in [3].

1 Sequences of Continuous Functions

Lemma 1.1 For any topological space $X$: If there is a well-ordered sequence of length $\xi$ in $C(X, \mathbb{R})$, then $\xi < d(X)^+$. 

Proof. Let $\{f_\alpha : \alpha < \xi\}$ be an increasing sequence in $C(X, \mathbb{R})$, and let $D \subseteq X$ be a dense subset of $X$ such that $d(X) = \max(|D|, \omega)$. By continuity, the $f_\alpha|D$ are all distinct; so, for each $\alpha < \xi$, choose a $d_\alpha \in D$ such that $f_\alpha(d_\alpha) < f_{\alpha+1}(d_\alpha)$. For each $d \in D$ the set $E_d = \{\alpha : d_\alpha = d\}$ is countable, because every well-ordered subset of $\mathbb{R}$ is countable. Since $\xi = \bigcup_{d \in D} E_d$, we have $|\xi| \leq \max(|D|, \omega) = d(X)$. \hfill $\square$

The converse implication is not true in general. For example, if $X$ has the countable chain condition (ccc), then every well-ordered chain in $C(X, \mathbb{R})$ is countable (because $X \times \mathbb{R}$ is also ccc). However, the converse is true for metric spaces:

Lemma 1.2 If $(X, \rho)$ is any non-empty metric space and $<$ is any total order of the cardinal $d(X)$, then there is a chain in $C(X, \mathbb{R})$ which is isomorphic to $\prec$. 

Proof. First, note that every countable total order is embeddable in \( \mathbb{R} \), so if \( d(X) = \omega \), then the result follows trivially using constant functions. In particular, we may assume that \( X \) is infinite, and then fix \( D \subseteq X \) which is dense and of size \( d(X) \). For each \( n \in \omega \), let \( D_n \) be a subset of \( D \) which is maximal with respect to the property \( \forall d, e \in D_n [d \neq e \rightarrow \rho(d, e) \geq 2^{-n}] \). Then \( \bigcup_n D_n \) is also dense, so we may assume that \( \bigcup_n D_n = D \). We may also assume that \( \prec \) is a total order of the set \( D \). Now, we shall produce \( f_d \in C(X, \mathbb{R}) \) for \( d \in D \) such that \( f_d < f_e \) whenever \( d < e \).

For each \( n \), if \( e \in D_n \), define \( \varphi^n_e(x) = \max(0, 2^{-n} - \rho(x, e)) \). For each \( d \in D \), let \( \psi^n_d = \sum \{ \varphi^n_c : c \in D_n \land c < d \} \). Since every \( x \in X \) has a neighborhood on which all but at most one of the \( \varphi^n_c \) vanish, we have \( \psi^n_d \in C(X, [0, 2^{-n}]) \), and \( \psi^n_d < \psi^n_e \) whenever \( d < e \). Thus, if we let \( f_d = \sum_{n<\omega} \psi^n_d \), we have \( f_d \in C(X, [0, 2]) \), and \( f_d \leq f_e \) whenever \( d < e \). But also, if \( d \in D_n \) and \( d < e \), then \( \psi^n_d(d) = 0 < 2^{-n} = \psi^n_e(d) \), so actually \( f_d < f_e \) whenever \( d < e \).

Putting these lemmas together, we have:

**Theorem 1.3** Let \((X, \rho)\) be a metric space. Then there exists a well-ordered sequence of length \( \xi \) in \( C(X, \mathbb{R}) \) iff \( \xi < d(X)^+ \).

**Corollary 1.4** A metric space \((X, \rho)\) is separable iff every well-ordered sequence in \( C(X, \mathbb{R}) \) is countable.

## 2 Sequences of Baire Class 1 Functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces, we can get well-ordered sequences of every type less than \( \omega_2 \). To prove this, we shall apply some basic facts about \( \subset^* \) on \( \mathcal{P}(\omega) \). As usual, for \( x, y \subseteq \omega \), we say that \( x \subseteq^* y \) iff \( x \setminus y \) is finite. Then \( x \subseteq^* y \) iff \( x \setminus y \) is finite and \( y \setminus x \) is infinite. This \( \subseteq^* \) partially orders \( \mathcal{P}(\omega) \).

**Lemma 2.1** If \( X \subset \mathcal{P}(\omega) \) is a chain in the order \( \subset^* \), then on \( X \) (viewed as a subset of the Cantor set \( 2^\omega \cong \mathcal{P}(\omega) \)), there is a chain of Baire class 1 functions which is isomorphic to \((X, \subseteq^*)\).

**Proof.** Note that for each \( x \in X \),

\[
\{ y \in X : y \subseteq^* x \} = \bigcup_{m \in \omega} \{ y \in X : \forall n \geq m \ [y(n) \leq x(n)] \}
\]
which is an $F_\sigma$ set in $X$. Likewise, the sets \( \{ y \in X : y \supseteq^* x \} \), \( \{ y \in X : y \subset^* x \} \), and \( \{ y \in X : y \supseteq^* x \} \), are all $F_\sigma$ sets in $X$, and hence also $G_\delta$ sets. It follows that if $f_x : X \to \{0,1\}$ is the characteristic function of \( \{ y \in X : y \subset^* x \} \), then $f_x : X \to \mathbb{R}$ is a Baire class 1 function. Then, \( \{ f_x : x \in X \} \) is the required chain.

\[ \square \]

**Lemma 2.2** For any infinite cardinal \( \kappa \), suppose that \((P(\omega), \subset^*)\) contains a chain \( \{ x_\alpha : \alpha < \kappa \} \) (i.e., \( \alpha < \beta \to x_\alpha \subset^* x_\beta \)). Then \((P(\omega), \subset^*)\) contains a chain $X$ of size $\kappa$ such that every ordinal \( \xi < \kappa^+ \) is embeddable into $X$.

**Proof.** Let $S = \bigcup_{1 \leq n < \omega} \kappa^n$. For $s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S$, let $s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1)$. Starting with the $x_\alpha = x_\alpha$, choose $x_s \in P(\omega)$ by induction on length\( (s) \) so that $x_s = x_{s^0} \subset^* x_{s^0} \subset^* x_{s^\beta} \subset^* x_{s^+}$ whenever $s \in S$ and $0 < \alpha < \beta < \kappa$. Let $X = \{ x_s : s \in S \}$. Then, whenever $x, y \in X$ with $x \subset^* y$, the ordinal $\kappa$ is embeddable in $(x, y) = \{ z \in X : x \subset^* z \subset^* y \}$. From this, one easily proves by induction on $\xi < \kappa^+$ (using $\text{cf}(\xi) \leq \kappa$) that $\xi$ is embeddable in each such interval $(x, y)$.

Since $P(\omega)$ certainly contains a chain of type $\omega_1$, these two lemmas yield:

**Theorem 2.3** There is a separable metric space $X$ on which, for every $\xi < \omega_2$, there is a well-ordered chain of length $\xi$ of Baire class 1 functions.

Under $CH$, this is best possible, since there will be only $2^\omega = \omega_1$ Baire class 1 functions on a separable metric space, so there could not be a chain of length $\omega_2$. Under $\neg CH$, the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with $c = 2^\omega$ being arbitrarily large that there is a chain in $(P(\omega), \subset^*)$ of type $c$; for example, this is true under $MA$ (see [2]). In this case, there will be a separable $X$ with well-ordered chains of all lengths less than $c^+$. However, in the Cohen model, where $c$ can also be made arbitrarily large, we never get chains of type $\omega_2$. We shall prove this by using the following lemma, which relates it to the rectangle problem:

**Lemma 2.4** Suppose that there is a separable metric space $Y$ with an $\omega_2$-chain of Borel subsets, \( \{ B_\alpha : \alpha < \omega_2 \} \) (so, $\alpha < \beta \to B_\alpha \subsetneq B_\beta$). Then in $\omega_2 \times \omega_2$, the well-order relation $<$ is in the $\sigma$-algebra generated by the set of all rectangles, \( \{ S \times T : S, T \in P(\omega_2) \} \).
Proof. Each $B_\alpha$ has some countable Borel rank. Since there are only $\omega_1$ ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say, each $B_\alpha$ is a $\Sigma^0_\mu$ set for some fixed $\mu < \omega_1$.

Let $J = \omega^\omega$, and let $A \subseteq Y \times J$ be a universal $\Sigma^0_\mu$ set; that is, $A$ is $\Sigma^0_\mu$ in $Y \times J$ and every $\Sigma^0_\mu$ subset of $Y$ is of the form $A^j = \{y : (y, j) \in A\}$ for some $j \in J$ (see [7], §31). Now, for $\alpha, \beta < \omega_2$, fix $y_\alpha \in B_{\alpha+1} \setminus B_\alpha$, and fix $j_\beta \in J$ such that $A^{j_\beta} = B_\beta$. Then $\alpha < \beta$ iff $(y_\alpha, j_\beta) \in A$. Thus, $\{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2\}$ is a Borel subset of $\{y_\alpha : \alpha < \omega_2\} \times \{j_\beta : \beta < \omega_2\}$, and is hence in the $\sigma$-algebra generated by open rectangles, so $\prec$, as a subset of $\omega_2 \times \omega_2$, is in the $\sigma$-algebra generated by rectangles.

\end{proof}

\begin{thm}
Assume that the well-order relation $\prec$ on $\omega_2$ is not in the $\sigma$-algebra generated by the set of all rectangles. Then no separable metric space can have a chain of length $\omega_2$ of Baire class 1 functions.
\end{thm}

\begin{proof}
Suppose that $\{f_\alpha : \alpha < \omega_2\}$ is a chain of Baire class one functions on the separable metric space $X$. Let $B_\alpha = \{(x, r) \in X \times \mathbb{R} : r \leq f_\alpha(x)\}$. Then the $B_\alpha$ form an $\omega_2$-chain of Borel subsets of the separable metric space $X \times \mathbb{R}$, so we have a contradiction by Lemma 2.4.
\end{proof}

Finally, we point out that the hypothesis of this theorem is consistent, since it holds in the extension $V[G]$ formed by adding $\geq \omega_2$ Cohen reals to a ground model $V$ which satisfies CH. This fact was first proved in [6]. It also follows from the more general principle $HP_2(\omega_2)$ of Brendle, Fuchino, and Soukup [1]. They define this principle, prove that it holds in Cohen extensions (and in a number of other forcing extensions), and show the following:

\begin{lem}
$HP_2(\kappa)$ implies that if $R$ is any relation on $\mathcal{P}(\omega)$ which is first-order definable over $H(\omega_1)$ from a fixed element of $H(\omega_1)$, then there is no $X \subseteq \mathcal{P}(\omega)$ such that $(X; R)$ is isomorphic to $(\kappa; \prec)$.
\end{lem}

These matters are also discussed in [4], which indicates how such statements are verified in Cohen extensions. Here, $H(\omega_1)$ denotes the set of hereditarily countable sets.

\begin{lem}
$HP_2(\omega_2)$ implies that in $\omega_2 \times \omega_2$, the well-order relation $\prec$ is not in the $\sigma$-algebra generated by the set of all rectangles, $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$.
\end{lem}

\begin{proof}
Suppose that $\prec$ were in this $\sigma$-algebra. Then we would have fixed $K_n \subseteq \omega_2$ for $n < \omega$ such that $\prec$ is in the $\sigma$-algebra generated by all the $K_m \times K_n$.
For each α, let $u_α = \{ n \in ω : α \in K_n \}$. There is then a formula $φ(x, y, z)$ and a fixed $w \in H(ω_1)$ such that for all $α, β < ω_2$, $α < β$ iff $H(ω_1) \models φ(u_α, u_β, w)$; here, $w$ encodes the particular countable boolean combination used to get $<$ from the $K_n$. Now, if $X = \{ u_α : α < ω_2 \}$, then $φ$ defines a relation $R$ on $H(ω_1)$ such that $(X; R)$ is isomorphic to $(ω_2; <)$, contradicting Lemma 2.6. □

References


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