SMALL LOCALLY COMPACT LINEARLY LINDELÖF SPACES

KENNETH KUNEN

Abstract. There is a locally compact Hausdorff space of weight ℵω which is linearly Lindelöf and not Lindelöf.

We shall prove:

**Theorem 1.** There is a compact Hausdorff space X and a point p in X such that:

1. \( \chi(p, X) = w(X) = \aleph_\omega \).
2. For all regular \( \kappa > \omega \), no \( \kappa \)-sequence of points distinct from p converges to p.

As usual, \( \chi(p, X) \), the character of p in X, is the least size of a local base at p, and \( w(X) \), the weight of X, is the least size of a base for X. This theorem with “\( \beth_\omega \)” replacing “\( \aleph_\omega \)” was proved in [11]. Arhangel’skii and Buzyakova [1] point out that if \( X, p \) satisfy (2) of the theorem, then the space \( X \setminus \{p\} \) is linearly Lindelöf and locally compact; if in addition \( \chi(p, X) > \aleph_0 \), then \( X \setminus \{p\} \) is not Lindelöf. (2) requires \( \text{cf}(\chi(p, X)) = \omega \), because there must be a sequence of type \( \text{cf}(\chi(p, X)) \) converging to p. Thus, in (1) of the theorem, \( \aleph_\omega \) is the smallest possible uncountable value for \( \chi(p, X) \) and \( w(X) \).

As in [11], the X of the theorem will be constructed as an inverse limit, using the following terminology:

**Definition 2.** An inverse system is a sequence \( \langle X_n, \pi_{n+1}^n : n \in \omega \rangle \), where each \( X_n \) is a compact Hausdorff space, and each \( \pi_{n+1}^n \) is a continuous map from \( X_{n+1} \) onto \( X_n \).

Such an inverse systems yields a compact Hausdorff space, \( X_\omega = \lim_{\rightarrow} X_n \), and maps \( \pi_m^\omega : X_\omega \rightarrow X_m \) for \( m < \omega \) and \( \pi_m^n : X_n \rightarrow X_m \) for \( m \leq n < \omega \). Exactly as in [11], one easily proves:

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Lemma 3. Suppose that \( \langle X_n, \pi_n^+ : n \in \omega \rangle \) is an inverse system and \( p \in X = X_\omega \), with the \( p_n = \pi_n^+(p) \in X_n \) satisfying:

A. Each \( p_n \) is a weak \( P_{\aleph_n} \)-point in \( X_n \).
B. Each \( w(X_n) < \aleph_\omega \).
C. Each \( (\pi_n^0)^{-1}\{p_0\} \) is nowhere dense in \( X_n \).

Then \( X, p \) satisfies Theorem 1.

As usual, \( y \in Y \) is a weak \( P_\kappa \)-point iff \( y \) is not in the closure of any subset of \( Y \setminus \{y\} \) of size less than \( \kappa \), and \( y \) is a \( P_\kappa \)-point iff the intersection of fewer than \( \kappa \) neighborhoods of \( y \) is always a neighborhood of \( y \). These properties are trivial for \( \kappa = \aleph_0 \). The terms “\( P \)-point” and “weak \( P \)-point” denote “\( P_{\aleph_1} \)-point” and “weak \( P_{\aleph_1} \)-point”, respectively.

Every \( P_\kappa \)-point is a weak \( P_\kappa \)-point, but as pointed out in [11], one cannot have each \( p_n \) being a \( P_{\aleph_n} \)-point, as that would contradict (C). In the construction we describe, it will be natural to make every \( p_n \) fail to be a \( P \)-point in \( X_n \).

We shall build the \( X_n \) and \( p_n \) inductively using the following:

Lemma 4. Assume that \( y \in F \subseteq Y \), where \( Y \) is compact Hausdorff, \( w(Y) \leq \aleph_n \), and \( \text{int}(F) = \emptyset \). Then there is a compact Hausdorff space \( X \), a point \( x \in X \), and a continuous \( g : X \to Y \) such that:

1. \( g(X) = Y \) and \( g(x) = y \).
2. \( g^{-1}(F) \) is nowhere dense in \( X \).
3. \( w(X) = \aleph_n \).
4. In \( X \), \( x \) is a weak \( P_{\aleph_n} \)-point and not a \( P \)-point.

Proof of Theorem 1. Inductively build an inverse system as in Lemma 3, with each \( w(X_n) = \aleph_n \). \( X_0 \) can be the Cantor set. When \( n > 0 \) and we are given \( X_{n-1}, p_{n-1} \), we apply Lemma 4 with \( F = (\pi_n^0)^{-1}\{p_0\} \). □

Of course, we still need to prove Lemma 4. We remark that we do not assume that \( F \) is closed, although that was true in our proof of Theorem 1. Even if \( F \) is dense in \( Y \) in Lemma 4, we still get (2) — that is, \( \text{int}(\text{cl}(g^{-1}(F))) = \emptyset \).

In Lemma 4, \( n \) can be 0, although this case is not used in the proof of Theorem 1. For this case, the “weak \( P_{\aleph_0} \)-point” is trivial, and the lemma is easily proved by an Aleksandrov duplicate construction. A more convoluted proof is: Let \( D \subseteq Y \setminus F \) be dense in \( Y \) and countable. Let \( g \) map \( \omega \) onto \( D \) and extend \( g \) to a map \( \beta g : \beta \omega \to Y \). Choosing \( x \) to be any point in \( (\beta g)^{-1}(\{y\}) \) yields (1)(2)(4), but \( \beta \omega \) has weight \( 2^{\aleph_0} \). Now, we can take a countable elementary submodel of the whole construction to get an \( X \) of weight \( \aleph_0 \). Our proof for a general \( n \) will follow this pattern.

As usual, \( \beta \kappa \) denotes the Čech compactification of a discrete \( \kappa \), and \( \kappa^* = \beta \kappa \setminus \kappa \). Equivalently, \( \beta \kappa \) is the space of ultrafilters on \( \kappa \), and \( \kappa^* \) is
the space of nonprincipal ultrafilters. If \( g : \kappa \to Y \), where \( Y \) is compact Hausdorff, then \( \beta g \) denotes the unique extension of \( g \) to a continuous map from \( \beta\kappa \) to \( Y \). Our weak \( P_\kappa \)-point in Lemma 4 will be a good ultrafilter in the sense of Keisler [9]:

**Definition 5.** An ultrafilter \( x \) on \( \kappa \) is good iff for all \( H : [\kappa]^{<\omega} \to x \), there is a \( K : \kappa \to x \) such that \( K(\alpha_1) \cap \cdots \cap K(\alpha_n) \subseteq H(s) \) for each \( s = \{\alpha_1, \ldots, \alpha_n\} \in [\kappa]^{<\omega} \).

The following is well-known:

**Lemma 6.** Let \( \kappa \) be any infinite cardinal.

1. There are ultrafilters \( x \) on \( \kappa \) which are both good and countably incomplete.
2. Any \( x \) as in (1) is a weak \( P_\kappa \) point and not a \( P \)-point in \( \beta\kappa \).

In (2), \( x \) is not a \( P \)-point by countable incompleteness, and proofs that it is a weak \( P_\kappa \) point can be found in [2, 3, 5]. For (1), see [4], Theorem 6.1.4; also, [2, 3] construct good ultrafilters with various additional properties.

We first point out (Lemma 9) that taking \( x \) to be a good ultrafilter on \( \omega_n \) will give us (1)(2)(4) of Lemma 4. Unfortunately, \( w(\beta\omega_n) = 2^{\aleph_0} \), so we shall take an elementary submodel to bring the weight down. Omitting the elementary submodel, our argument is as in [11], which obtained the \( X \) of Theorem 1 with \( w(X) = \aleph_1 \), rather than \( \aleph_\omega \). A related use of elementary submodels to reduce the weight occurs in [7].

Before we consider the weight problem, we explain how to map the good ultrafilter onto the given point \( y \). This part of the argument works for any regular ultrafilter.

**Definition 7.** An ultrafilter \( x \) on \( \kappa \) is regular iff there are \( E_\alpha \in x \) for \( \alpha < \kappa \) such that \( \{\alpha : \xi \in E_\alpha\} \) is finite for all \( \xi < \kappa \).

Such an \( x \) is countably incomplete because \( \bigcap_{n<\omega} E_n = \emptyset \). For the following, see Exercise 6.1.3 of [4] or the proof of Lemma 2.1 in Keisler [10]:

**Lemma 8.** If \( x \) is a countably incomplete good ultrafilter on \( \kappa \), then \( x \) is regular.

**Lemma 9.** Let \( x \) be a regular ultrafilter on \( \kappa \). Assume that \( y \in F \subseteq Y \), where \( Y \) is compact Hausdorff, \( w(Y) \leq \kappa \), and \( \text{int}(F) = \emptyset \). Then there is a map \( g : \kappa \to Y \) such that:

A. \( \beta g \) maps \( \beta\kappa \) onto \( Y \).
B. \( (\beta g)(x) = y \).
C. \( g(\xi) \notin F \) for all \( \xi \in \kappa \).
D. \( g^{-1}(F) \) is nowhere dense in \( \beta\kappa \).
Proof. Of course, (D) follows from (C) because \( g^{-1}(F) \subseteq \kappa^* \). Fix \( A \subseteq \kappa \) with \( A \notin x \) and \( |A| = \kappa \). Let \( \{ E_\alpha : \alpha < \kappa \} \) be as in Definition 7, with each \( E_\alpha \cap A = \emptyset \). Let \( \{ U_\alpha : \alpha < \kappa \} \) be an open base at \( y \) in \( Y \). Let \( D \subseteq Y \setminus F \) be dense in \( Y \) with \( |D| \leq \kappa \). Choose \( g : \kappa \rightarrow Y \) such that \( g \) maps \( A \) onto \( D \) (ensuring (A)) and each \( g(\xi) \in \bigcap\{ U_\alpha : \xi \in E_\alpha \} \setminus F \) (ensuring (B)(C)). \( \square \)

To apply the elementary submodel technique (as in Dow [6]), we put the construction of Lemma 9 inside an \( H(\theta) \), where \( \theta \) is a suitably large regular cardinal. Let \( M < H(\theta) \), with \( \kappa \subseteq M \) and \( |M| = \kappa \), such that \( M \) contains \( Y \) and its topology \( T \), along with \( F, g, x, y \). Let \( B = \mathcal{P}(\kappa) \cap M \), let \( \text{st}(\mathcal{B}) \) denote its Stone space, and let \( \Gamma : \beta\kappa \rightarrow \text{st}(\mathcal{B}) \) be the natural map; so \( \Gamma(x) = x \cap B = x \cap M \). Since \( T \cap M \) is a base for \( Y \) (by \( w(Y) \leq \kappa \)), we have \( \Gamma(z_1) = \Gamma(z_2) \rightarrow (\beta g)(z_1) = (\beta g)(z_2) \), so that \( \beta g \) yields a map \( \tilde{g} : \text{st}(\mathcal{B}) \rightarrow Y \) with \( \beta g = \tilde{g} \circ \Gamma \). Note that \( \mathcal{B} \) contains all finite subsets of \( \kappa \), so that \( \text{st}(\mathcal{B}) \) is some compactification of a discrete \( \kappa \). It is easily seen that we still have (A–D), replacing \( \beta g \) by \( \tilde{g} \), \( \beta \kappa \) by \( \text{st}(\mathcal{B}) \), and \( x \) by \( \Gamma(x) \). Note that \( \Gamma(x) \) must be countably incomplete by \( M < H(\theta) \), so that \( \Gamma(x) \) will not be a \( P \)-point in \( \text{st}(\mathcal{B}) \). But to prove Lemma 4 (letting \( \kappa = \aleph_n \)), we also need \( \Gamma(x) \) to be a weak \( P_\kappa \)-point in \( \text{st}(\mathcal{B}) \). We may assume that \( x \in \beta\kappa \) is good, so it is a weak \( P_\kappa \)-point there. But we need to show that in \( \text{st}(\mathcal{B}) \), \( \Gamma(x) \) is not a limit point of any set of size \( \lambda < \kappa \). Our argument here needs to assume that \( M \) is \( \lambda \)-covering and that \( \lambda^+ \) is not a Jónsson cardinal. These two assumptions will cause no problems when \( \lambda < \aleph_\omega \).

As usual, \( M < H(\theta) \) is \( \lambda \)-covering iff for all \( E \in [M]^\lambda \), there is an \( F \in [M]^\theta \) such that \( E \subseteq F \) and \( F \in M \). By taking a union of an elementary chain of type \( \lambda^+ \) (see [6], §3), we see that there is an \( M < H(\theta) \) with \( |M| = \lambda^+ \) such that \( M \) is \( \lambda \)-covering.

\( \kappa \) is called a Jónsson cardinal iff for all \( \psi : [\kappa]^{<\omega} \rightarrow \kappa \), there is a \( W \in [\kappa]^\kappa \) such that \( \psi(W)^{<\omega} \) is a proper subset of \( \kappa \). By Tryba [12] (or see [8]):

Lemma 10. No successor to a regular cardinal is Jónsson.

In particular, each \( \aleph_n \) is not a Jónsson cardinal; this fact is much older and is easily proved by induction on \( n \).

Lemma 11. Let \( \kappa \) be infinite and \( x \in \beta\kappa \) a good ultrafilter on \( \kappa \). Fix an infinite \( \lambda < \kappa \) and let \( \theta > 2^\lambda \) be regular. Let \( M < H(\theta) \), with \( x, \kappa \in M \) and \( \kappa \subseteq M \). Assume that \( M \) is \( \lambda \)-covering and \( \lambda^+ \) is not a Jónsson cardinal. Let \( B = \mathcal{P}(\kappa) \cap M \), and let \( \Gamma : \beta\kappa \rightarrow \text{st}(\mathcal{B}) \) be the natural map. Then \( \Gamma(x) \) is a weak \( P_{\lambda^+} \)-point of \( \text{st}(\mathcal{B}) \).

Proof. Fix \( Z \subseteq \text{st}(\mathcal{B}) \setminus \{ \Gamma(x) \} \) with \( |Z| \leq \lambda \). We shall show that \( \Gamma(x) \) is not in the closure of \( Z \). For each \( z \in Z \), choose \( F_z \in \Gamma(x) = x \cap B = x \cap M \) such that \( F_z \notin z \). Since \( M \) is \( \lambda \)-covering, we can get \( \langle G_\xi : \xi < \lambda \rangle \in M \) such
that each \( G_\xi \in x \) and \( \forall z \in Z \exists \xi < \lambda [G_\xi = F_z] \). Since \( \lambda^+ \) is not Jónsson and \( \lambda^+ \in M \), we can fix \( \psi \in M \) such that \( \psi : [\lambda^+]^{<\omega} \to \lambda \) and such that 
\[
\psi([W]^{<\omega}) = \lambda \text{ for all } W \in [\lambda^+]^{\lambda^+}.
\]
Define \( H(s) = G_{\psi(s)} \). Then \( H \in M \) and \( H : [\lambda^+]^{<\omega} \to \Gamma(x) \). Since \( x \) is good, we can find \( \langle K_\alpha : \alpha < \lambda^+ \rangle \in M \) such that \( K_\alpha \) is in \( x \) and such that \( K_\alpha \cap \cdots \cap K_\alpha \subseteq H(\{\alpha_1, \ldots, \alpha_n\}) \) for each \( n \) and each \( \alpha_1, \ldots, \alpha_n \in \lambda^+ \).

Now (in \( V \)), we claim that \( \exists \alpha < \lambda^+ \forall z \in Z [K_\alpha \notin z] \) (so that \( \Gamma(x) \notin \text{cl}(Z) \)). If not, then we can fix \( W \in [\lambda^+]^{\lambda^+} \) and \( z \in Z \) such that \( K_\alpha \in z \) for all \( \alpha \in W \). Fix \( \xi < \lambda \) such that \( G_\xi = F_z \). Since \( \psi([W]^{<\omega}) = \lambda \), fix \( s \in [W]^{<\omega} \) such that \( \psi(s) = \xi \). Say \( s = \{\alpha_1, \ldots, \alpha_n\} \). Then \( G_\xi = G_{\psi(s)} = H(s) \supseteq K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subseteq z \), a contradiction, since \( F_z \notin z \).

\section*{Proof of Lemma 4.} Use Lemmas 11 and 9, with \( \kappa = \lambda^+ = \aleph_n \).

In view of Lemma 10, we can also prove Theorem 1 replacing \( \aleph_\omega \) with any other singular cardinal of cofinality \( \omega \), since we can replace \( \aleph_n \) in Lemma 4 by any successor to a regular cardinal.

\section*{References}


\textsc{Department of Mathematics, University of Wisconsin, Madison, WI 57306 USA}

\textit{E-mail address:} kunen@math.wisc.edu

\textit{URL:} \url{http://www.math.wisc.edu/~kunen/}