LIMITS IN THE UNIFORM ULTRAFILTERS

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ABSTRACT. Let $u(\kappa)$ be the space of uniform ultrafilters on $\kappa$. If $\kappa$ is
regular, then there is an $x \in u(\kappa)$ which is not an accumulation point
of any subset of $u(\kappa)$ of size $\kappa$ or less. $x$ is also good, in the sense of
Keisler.

1. Introduction

If $\kappa$ is any infinite cardinal, we have $u(\kappa) \subseteq \kappa^* \subseteq \beta \kappa$. Here, $\kappa$ has the
discrete topology, so its Čech compactification, $\beta \kappa$, is the space of ultrafilters
on $\kappa$, and $\kappa^* = \beta \kappa \setminus \kappa$ is the the space of nonprincipal ultrafilters on $\kappa$. Then,
we use $u(\kappa)$ to denote the space of uniform ultrafilters; that is, $x \in u(\kappa)$ iff
every set in $x$ has size $\kappa$.

In studying limits, one is naturally led to $P$-points and weak $P$-points:

Definition 1.1. If $\theta$ is an infinite cardinal and $x$ is a point in a topological
space $X$, then:

$\blacklozenge$ $x$ is a $P_\theta$-point in $X$ iff the intersection of any family of fewer than $\theta$
neighborhoods of $x$ is also a neighborhood of $x$.

$\blacksquare$ $x$ is a weak $P_\theta$-point in $X$ iff $x$ is not a limit point of any subset of
$X \setminus \{x\}$ of size less than $\theta$.

So, a $P$-point is a $P_{\omega_1}$-point, and a weak $P$-point is a weak $P_{\omega_1}$-point. In
any $T_1$ space, every $P_{\omega_1}$-point is a weak $P_{\theta}$-point. The main theorem of this
paper is:

Theorem 1.2. If $\kappa$ is regular, then there is an $x \in u(\kappa)$ which is a weak
$P_{\kappa^*}$-point in $u(\kappa)$, and hence a weak $P_{\kappa}$-point in $\beta \kappa$.

This is best possible as a ZFC result. Note that for $x \in X \subseteq Y$, if $x$ is
a weak $P_{\theta}$-point in $Y$, then $x$ is a weak $P_{\theta}$-point in $X$; but not conversely,
in general. If $x \in u(\kappa)$, it cannot be a weak $P_{\kappa^*}$-point in $\beta \kappa$, or even in $\kappa^*$
if $2^{< \kappa} = \kappa$; and, $x$ cannot be a weak $P_{\kappa^+}$-point in $u(\kappa)$ if $2^\kappa = \kappa^+$. The
“hence” in the theorem is correct because by regularity of $\kappa$, no $x \in u(\kappa)$
can be a limit of a subset of $\beta \kappa \setminus u(\kappa)$ of size less than $\kappa$.

For $\kappa = \omega$ (where $u(\omega) = \omega^*$), the theorem is already known (Kunen
[5]). By a still earlier result of W. Rudin [6], it is consistent with ZFC that
there is even a $P$-point in $\omega^*$, although the existence of $P$-points is also independent (Shelah). The theorem for $\kappa = \omega$ can be improved in various models of $\neg CH$. For example, if $MA$ holds, then Rudin's proof yields a $P$-point (and hence a weak $P$-point) in $\omega^*$. However, it is consistent with $\tau$ arbitrarily large that there are no weak $P_{\omega^2}$-points in $\omega^*$ (add random reals or Cohen reals to a model of $CH$).

For $\kappa > \omega$, we do not know about the consistency of possible strengthenings of this theorem (along the $\kappa = \omega$ lines), except for the following two elementary remarks: First, if $\kappa$ is regular and larger than $\omega$ but less than the first (2-valued) measurable cardinal, then there are no $P$-points in $u(\kappa)$. Second, it is consistent with $2^\kappa \gg \kappa^+$ that there are no weak $P_{\kappa^+}$-points in $u(\kappa)$; just use Cohen forcing (with conditions of size less than $\kappa$) over a model of $GCH$.

We do not know if the theorem holds for singular $\kappa$. However, the existence of a weak $P_\kappa$-point in $\beta \kappa$ is already known, and this holds for singular $\kappa$ as well. In fact, it is well-known (see Section 2) that if $x \in u(\kappa)$ is $\kappa^+$-good in the sense of Keisler, then $x$ is a weak $P_\kappa$-point in $\beta \kappa$, and hence also in $u(\kappa)$. Such good ultrafilters exist (Keisler [3] under $GCH$ and Kunen [4] in ZFC). The $x$ in Theorem 1.2 will be $\kappa^+$-good, and we shall add another ingredient to the inductive construction of [4] to make it also a weak $P_{\kappa^+}$-point in $u(\kappa)$. Not every $\kappa^+$-good ultrafilter is a weak $P_{\kappa^+}$-point in $u(\kappa)$; for a counter-example, use the product of two good ultrafilters.

In Section 2, we first observe that the notion “good” is really a topological notion which makes sense for a point $x$ in an arbitrary space $X$. Every $\kappa^{++}$-good point is a weak $P_{\kappa^{++}}$-point. However, one cannot prove in ZFC that there is an $x \in u(\kappa)$ which is $\kappa^{++}$-good in $u(\kappa)$ (Theorem 2.8). So, we shall weaken “$\kappa^{++}$-good” to “$\kappa^+$-mediocre”. Every point which is both $\kappa^+$-mediocre and $\kappa^+$-good is a weak $P_{\kappa^+}$-point. The rest of the paper is then devoted to showing that such points exist in $u(\kappa)$.

The proof follows the standard pattern from [4, 5]; one constructs an ultrafilter in $2^\kappa$ steps with the aid of a matrix of sets. However, the matrix used here is a little more complicated than the standard ones, and it requires some argument to prove that it really exists. Section 3 proves some general results on constructing matrices, and Section 4 constructs the actual matrix we need. Then, Section 6 uses this matrix to prove Theorem 1.2. Regularity of $\kappa$ is used only to prove that the matrix exists; the construction of the ultrafilter given the matrix works for any $\kappa$.

Notation: When discussing subsets of $\kappa$, $A \subseteq^* B$ means that $|A \setminus B| < \kappa$, $A =^* B$ means that $A \subseteq^* B \subseteq^* A$, and $A \perp B$ means that $|A \cap B| < \kappa$. $\mathcal{P}(\kappa)/{<\kappa}$ denotes the quotient algebra of $\mathcal{P}(\kappa)$ modulo the ideal $\{A \in \mathcal{P}(\kappa) : |A| < \kappa\}$. Then $[A] \in \mathcal{P}(\kappa)/{<\kappa}$ is the equivalence class of $A \in \mathcal{P}(\kappa)$. For $\kappa = \omega$, we use $\mathcal{P}(\omega)/fin$ for $\mathcal{P}(\omega)/{<\omega}$.

2. Good and Mediocre Points

The following notation for intersections will be used throughout this paper:
Definition 2.1. Given sets $X_\alpha$ for $\alpha < \theta$ and $p \in [\theta]^{< \omega}$: $X_p = \bigcap_{\alpha \in p} X_\alpha$.

Definition 2.2. A point $x$ in the topological space $X$ is $\theta^+$-good iff, given neighborhoods $U_\alpha (r \in [\theta]^{< \omega})$ of $x$, there are neighborhoods $V_\alpha (\alpha < \theta)$ of $x$ such that $V_\alpha \subseteq U_r$ for each non-empty $r \in [\theta]^{< \omega}$.

$V_\emptyset$ will play no role in our arguments, but it would be natural to let it be $X$ when we are discussing subsets of $X$.

Definition 2.3. A point $x$ in the space $X$ is $\theta^+$-mediocre iff, for some fixed collection of one-to-one functions $\varphi_\beta : \beta \to \theta$ ($\beta < \theta^+$), are neighborhoods of $x$, there are neighborhoods $V_\alpha (\alpha < \theta^+)$ of $x$ such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$ whenever $\alpha < \beta < \theta^+$.

Every $\omega_1$-mediocre point is $\omega_1$-OK (in the sense of [5]), but past $\omega_1$, the notions of “mediocre” and “OK” diverge.

Lemma 2.4. Every $\lambda^{++}$-good point is $\lambda^+$-mediocre.

Proof. Fix a collection of one-to-one functions $\varphi_\beta : \beta \to \lambda$ ($\beta < \lambda^+$). Let $U_\xi (\xi \in \lambda)$ be any neighborhoods of the point $x$. Now define new neighborhoods $U_\xi^* (r \in [\lambda^+]^{< \omega})$ of $x$ by setting $U_\xi^* = U_{\varphi_\beta(\alpha)}$ whenever $\alpha < \beta < \lambda^+$, and setting all other $U_\xi^* = X$. Apply $\lambda^{++}$-goodness to get $V_\alpha (\alpha < \lambda^+)$, then $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$.

It is a folklore result that every $\kappa^+$-good point in a $T_1$ space is a weak $P_\kappa$-point (see, e.g., Dow [1]). Examining that proof, we see that in fact mediocrity suffices:

Lemma 2.5. If $x \in X$ is a weak $P_\kappa$-point which is also $\kappa^+$-mediocre, then $x$ is a weak $P_{\kappa^+}$-point.

Proof. Let $\varphi_\beta : \beta \to \kappa$ ($\beta < \kappa^+$) be as in Definition 2.3. Given $Y = \{y_\xi : \xi \in \kappa\} \subseteq X \setminus \{x\}$, we wish to show that $x$ is not in the closure of $Y$. For $\xi \in \kappa$, let $U_\xi$ be a neighborhood of $x$ which misses $\{y_\eta : \eta \leq \xi\}$. Now, fix neighborhoods $V_\alpha (\alpha < \kappa^+)$ of $x$ such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$.

Then for some $\alpha$, $V_\alpha \cap Y = \emptyset$: If not, then we can find $\xi \in \kappa$ and $E \subseteq \kappa$ with $|E| = \kappa^+$ such that $\forall \alpha \in E \ (y_\xi \in V_\alpha)$. Let $\beta$ be any element of $E$ such that $|E \cap \beta| = \kappa$. So, $\varphi_\beta(E \cap \beta)$ is unbounded in $\kappa$. Choose $\alpha \in E \cap \beta$ such that $\varphi_\beta(\alpha) > \xi$. Then $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$ which misses $\{y_\eta : \eta \leq \varphi_\beta(\alpha)\}$. So $y_\xi \not\in V_\alpha \cap V_\beta$, but this is a contradiction, since $\alpha, \beta \in E$.

Lemma 2.6. If $x$ is $\kappa^+$-good in the $T_1$ space $X$, then $x$ is a weak $P_\kappa$-point in $X$.

Proof. Observe that for all $\lambda < \kappa$: $x$ is $\lambda^{++}$-good, and hence (by Lemma 2.4) $\lambda^+$-mediocre. Now, use Lemma 2.5, and show, by induction on $\lambda < \kappa$, that $x$ is a weak $P_{\lambda^+}$-point for all $\lambda < \kappa$.

Clearly, if $x \in X \subseteq Y$ and $x$ is $\kappa^+$-good in $Y$, then $x$ is $\kappa^+$-good in $X$; and there are trivial examples to show that the converse is false. However,
Lemma 2.7. If $\kappa$ is regular and $x \in u(\kappa)$ is $\kappa^+$-good in $u(\kappa)$, then $x$ is $\kappa^+$-good in $\beta\kappa$.

Proof. Viewing $x$ as an ultrafilter on $\kappa$, we prove $\kappa^+$-goodness in $\beta\kappa$ by fixing $U_r \in x$ for $r \in [\kappa]^{<\omega}$ and producing $V_\xi \in x$ for $\xi < \kappa$ such that each $V_\xi \subseteq U_r$. Now, $\kappa^+$-goodness in $u(\kappa)$ only gives us $W_\xi \in x$ such that each $W_\xi \subseteq^* U_r$. So, define $V_\xi = W_\xi \setminus \bigcup\{W_\eta : \max(r) = \xi\}$. Then each $V_\xi \subseteq U_r$. Each $V_\xi =^* W_\xi$ (since $\kappa$ is regular), so each $V_\xi \in x$ (since $x$ is uniform).

We shall prove Theorem 1.2 by producing an $x \in u(\kappa)$ which is both $\kappa^+$-good and $\kappa^+$-mediocre in $u(\kappa)$ (and then applying Lemmas 2.6 and 2.5), but then $x$ will also be $\kappa^+$-good in $\beta\kappa$ (the usual sense of "good" in model theory). An obvious improvement of this result would be an ultrafilter which is $\kappa^++$-good in $u(\kappa)$, but one cannot produce that in ZFC.

Theorem 2.8. If $\kappa$ is regular, $2^\kappa = \kappa^+$, and $\kappa > \omega$, then no countably incomplete $x \in u(\kappa)$ is $\kappa^+$-good in $u(\kappa)$.

Proof. Choose $A_n^\alpha \in x$ for $n < \omega < \alpha < \kappa$ so that every $\omega$-sequence of elements of $x$ is of the form $\langle A_n^\alpha \in x : n < \omega \rangle$ for some $\alpha$. Then choose $B_n^\alpha \in x$ so that $B_n^\alpha \subseteq A_n^\alpha$, $\bigcap_n B_n^\alpha = \emptyset$, and $B_0^\alpha \supseteq B_1^\alpha \supseteq B_2^\alpha \supseteq \cdots$. If $x$ were $\kappa^+$-good in $u(\kappa)$, we could choose $D_n \in x$ for $\alpha < \kappa$ so that $D_n \cap D_n^\alpha \subseteq^* B_{n+1}^\alpha$ whenever $n < \omega < \alpha < \kappa$. Now, fix $\alpha > \omega$ such that each $D_n = A_n^\alpha$. Then each $D_n \cap B_n^\alpha \subseteq D_n \cap A_n^\alpha = D_n \cap D_n \subseteq^* B_{n+1}^\alpha$, so $D_n \perp B_n \setminus B_{n+1}^\alpha$, and hence $D_n \perp \bigcup_n (B_n \setminus B_{n+1}^\alpha) = B_0^\alpha$, a contradiction, since $D_n, B_0^\alpha \in x$.

Note that this theorem fails for $\kappa = \omega$, since every $P$-point is $\omega_2$-good. Likewise, if $\kappa$ is measurable, then every normal ultrafilter on $\kappa$ is a $P_{\kappa^+}$-point, and hence $\kappa^{++}$-good, in $u(\kappa)$.

3. Matrices

Ultrafilters on $\kappa$ are often constructed with the aid of a matrix of sets consisting of $2^\kappa$ independent rows, each row being an instance of some base matrix, $M$. Since $M$ is sometimes a bit complicated, it may help to give an abstract discussion of such matrices:

Definition 3.1. An abstract matrix is a triple, $M = (B, \mathcal{J}, \mathcal{P})$, such that $B$ is a boolean algebra, $\mathcal{J}$ is an ideal in $B$, and $\mathcal{P} \subseteq B \setminus \mathcal{J}$. A $\theta \times M$ independent matrix in a boolean algebra $A$ is a sequence $\langle h_\alpha : \alpha < \theta \rangle$, where each $h_\alpha : B \to A$ is a homomorphism, $h_\alpha(b) = 0$ whenever $b \in \mathcal{J}$, and $\bigwedge_{\alpha \leq n} h_\alpha(b_i) \neq 0$ whenever $n \in \omega$, each $b_0, \ldots, b_n \in \mathcal{P}$, and the $\alpha_0, \ldots, \alpha_n < \theta$ are distinct.

Of course, one can always reduce this to the case $\mathcal{J} = \{0\}$ by replacing $B$ by $B \setminus \mathcal{J}$, but the definition as stated is closer to the way matrices are specified in practice, where it is simpler to take $B$ to be a free algebra. For example, say we want to specify $\lambda$ disjoint elements. Let $B$ be the free
algebra generated by \( \{b_\alpha : \alpha < \lambda\} \). Let \( \mathcal{J} \) be generated by all \( b_\alpha \land b_\beta \) for \( \alpha < \beta < \lambda \). Let \( \mathcal{P} \) (the positive elements) be \( \{b_\alpha : \alpha < \lambda\} \). Then a \( \theta \times \mathcal{M} \) independent matrix in the boolean algebra \( \mathcal{P}(\kappa) / < \kappa \) consists of \( \theta \) independent copies of \( \lambda \) almost disjoint sets. With this \( \mathcal{M} \), taking \( \lambda = \kappa^+ \) and \( \kappa \) regular, Dow [1] showed that there is a \( 2^\kappa \times \mathcal{M} \) independent matrix in \( \mathcal{P}(\kappa) / < \kappa \). Generalizing this, we show, for any \( \mathcal{M} \), how to construct a \( 2^\kappa \times \mathcal{M} \) matrix if we are given \( m \times \mathcal{M} \) matrices for each \( m \in \omega \).

**Definition 3.2.** \( L = L_\kappa = \{\gamma < \kappa : \gamma = 0 \text{ or } \gamma \text{ is a limit}\} \). \( \mathcal{F} \subseteq \omega^\kappa \) is mildly independent iff \( f(\gamma + m) \leq m \) for all \( \gamma \in L, m \in \omega, f \in \mathcal{F} \), and whenever \( n < \omega \) and \( f_0, \ldots, f_\kappa \in \mathcal{F} \) are all distinct, there is a \( \xi < \kappa \) such that \( f_0(\xi), \ldots, f_\kappa(\xi) \) are all distinct.

**Lemma 3.3.** For any infinite \( \kappa \), there is a mildly independent \( \mathcal{F} \subseteq \omega^\kappa \) with \( |\mathcal{F}| = 2^\kappa \).

**Proof.** Fix \( \mathcal{G} \subseteq \omega^\kappa \) which is independent in the usual sense with \( |\mathcal{G}| = 2^\kappa \) (Engelking and Karlowicz [2]). So, for all distinct \( g_0, \ldots, g_\kappa \in \mathcal{G} \) and all \( i_0, \ldots, i_\kappa \in \omega \), there are \( \kappa \) different \( \alpha < \kappa \) such that \( g_i(\alpha) = i_\ell \) for \( \ell = 0, \ldots, \kappa \).

If \( \kappa > \omega \), then fix \( g \in \mathcal{G} \). Each \( |g^{-1}\{\alpha\}| = \kappa = |L| \), so by permuting the index set \( \kappa \), we may assume that \( g(\gamma + m) = m \) for all \( \gamma \in L \) and \( m \in \omega \). Now, let \( \mathcal{F} = \{f \land g : f \in \mathcal{G}\} \), where \( (f \land g)(\xi) = \min(f(\xi), g(\xi)) \).

If \( \kappa = \omega \), let \( I(m) = m \) for each \( m \), and let \( \mathcal{F} = \{f \land I : f \in \mathcal{G}\} \). \( \square \)

**Theorem 3.4.** Assume that \( \kappa \) is regular and uncountable and let \( \mathcal{M} = (\mathcal{B}, \mathcal{J}, \mathcal{P}) \) be an abstract matrix. Assume that for each \( m \in \omega \), there is an \( m \times \mathcal{M} \) independent matrix in \( \mathcal{P}(\kappa) / < \kappa \). Then there is a \( 2^\kappa \times \mathcal{M} \) independent matrix in \( \mathcal{P}(\kappa) / < \kappa \).

**Proof.** To simplify the notation, assume that \( \mathcal{J} = \{0\} \) (replace \( \mathcal{B} \) by \( \mathcal{B}/\mathcal{J} \)).

For each \( m \), let \( \{h^m_j : j \leq m\} \) be an \( (m + 1) \times \mathcal{M} \) independent matrix. Choose \( H^m_j : \mathcal{B} \rightarrow \mathcal{P}(\kappa) \) so that \( h^m_j(b) = [H^m_j(b)] \) and \( H^m_j(b) = \kappa \cdot H^m_j(b) \); these \( H^m_j \) need not be homomorphisms. Let \( \mathcal{F} \subseteq \omega^\kappa \) be mildly independent with \( |\mathcal{F}| = 2^\kappa \). Let \( D = \{(\xi, \eta) : \xi \leq \eta < \kappa\} \). Then \( |D| = \kappa \). Define \( k_f : \mathcal{B} \rightarrow \mathcal{P}(\mathcal{D}) / < \kappa \) by:

\[
\begin{align*}
&k_f(b) = \left[D \cap \bigcup \left\{\{\gamma + m\} \times H^m_{f(\gamma + m)}(b) : \gamma \in L, m \in \omega\}\right]\right]
\end{align*}
\]

Fix \( f \); we shall show that \( k_f \) is a homomorphism. It preserves complements by our choice of the \( H^m_j \), so fix \( b_1, b_2, b_3 \in \mathcal{B} \) with \( b_3 = b_1 \land b_2 \). For each \( m \) and \( j \), \( H^m_j(b_3) = \ast H^m_j(b_1) \cap H^m_j(b_2) \), so since \( \kappa > \omega \), there is a fixed \( \zeta < \kappa \) such that \( H^m_j(b_3) \Delta (H^m_j(b_1) \cap H^m_j(b_2)) \subseteq \zeta \) for every \( m, j \). It follows that in \( \mathcal{P}(\mathcal{D}) / < \kappa \), \( k_f(b_3) \Delta (k_f(b_1) \land k_f(b_2)) \leq [D \cap (\kappa \times \zeta)] = 0 \).

To prove independence of the \( k_f \), fix non-zero \( b_0, \ldots, b_\kappa \in \mathcal{P} \) and distinct \( f_0, \ldots, f_\kappa \in \mathcal{F} \). By mild independence of \( \mathcal{F} \), fix \( \gamma \in L \) and \( m \in \omega \) such that \( f_0(\gamma + m), \ldots, f_\kappa(\gamma + m) \) are all distinct. Then \( \bigcap_{\kappa \leq n} k_f(b_\kappa) \geq \left[[\gamma + m] \times \bigcap_{\kappa \leq n} H^m_{f(\gamma + m)}(b_\kappa)\right] > 0 \). \( \square \)
We do not know whether this theorem holds for \( \kappa = \omega \), although it does hold in the case that \( |B| \leq \omega_1 \), in which case one can drop the hypothesis on the \( m \times \mathcal{M} \) independent matrices. First:

**Lemma 3.5.** Suppose that \( |B| \leq \omega_1 \) and \( h_j \) is a homomorphism from \( B \) into \( \mathcal{P}(\omega,f) \) for each \( j < \omega \). Then one can choose \( H_j : B \to \mathcal{P}(\omega) \) such that each \( h_j(b) = [H_j(b)] \), each \( H_j(B) = \mathcal{P}(\omega) \), and for each finite subalgebra \( A \subseteq B \), \( H_j \downarrow A \) is a homomorphism from \( A \) into \( \mathcal{P}(\omega) \) for all but finitely many \( j \).

**Proof.** Write \( B \) as a continuous increasing union of countable boolean algebras, \( B = \bigcup_{\alpha < \omega} B_\alpha \). By induction on \( \alpha \), define all the \( H_j \downarrow B_\alpha \). Let \( B_0 = \{0,1\} \), and let \( H_j(0) = \emptyset \) and \( H_j(1) = \omega \). Since there is no problem at limits, we can fix \( \alpha \) and explain how to extend the \( H_j \) to \( B_{\alpha+1} \). Now, write \( B_{\alpha+1} \) as an increasing union of finite boolean algebras, \( B_{\alpha+1} = \bigcup_{n < \omega} A_n \), with \( A_0 = \{0,1\} \). We may assume that we have \( n_0 \leq n_1 \leq n_2 \leq \cdots \) with \( \sup_n n_j = \omega \) and each \( H_j \downarrow (A_{n_j} \cap B_\alpha) \) a homomorphism. Since \( A_{n_j} \) is finite, one can extend \( H_j \downarrow (A_{n_j} \cap B_\alpha) \) to \( A_{n_j} \) so that it is still a homomorphism. Now that we have \( H_j \) defined on \( B_\alpha \cup A_{n_j} \), we can extend it to all of \( B_{\alpha+1} \) by the Axiom of Choice. \( \square \)

**Theorem 3.6.** Let \( \mathcal{M} = (B,\mathcal{J},\mathcal{P}) \) be an abstract matrix, with \( |B| \leq \omega_1 \). Then there is a \( 2^\omega \times \mathcal{M} \) independent matrix in \( \mathcal{P}(\omega,f) \).

**Proof.** Let the \( h_j^m : B \to \mathcal{P}(\omega,f) \) be as in the proof of Theorem 3.4. This is possible because every boolean algebra of size \( \omega_1 \) (in particular, the direct sum of \( m \) copies of \( B/\mathcal{J} \)) can be embedded into \( \mathcal{P}(\omega,f) \).

Now, follow the proof of Theorem 3.4, but choose the \( H_j^m : B \to \mathcal{P}(\kappa) \) as in Lemma 3.5, so that whenever \( b_3 = b_1 \wedge b_2 \), we have \( H_j^m(b_3) = H_j^m(b_1) \cap H_j^m(b_2) \) for all but finitely many \( m,j \). Define the \( k_f \) in precisely the same way, although \( D \) can also be \( \omega \times \omega \) now. \( \square \)

4. THE HAT TRICK

**Definition 4.1.** A hat function is a function \( \widehat{\phi} : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega} \) satisfying:

- \( \forall \alpha, p \subseteq q \Rightarrow \widehat{\phi}(p) \subseteq \widehat{\phi}(q) \).
- \( \forall \emptyset = \emptyset \).

If \( \phi_\beta : \beta \to \kappa \) is one-to-one for each \( \beta < \kappa^+ \), the derived hat function is defined by: \( \widehat{\phi} = \{ \phi_\beta(\alpha) : \alpha, \beta \in p \wedge \alpha < \beta \} \).

We have three goals in introducing this notion: First, to simplify the construction of a mediocream point by referring to \( \widehat{\phi} \) rather than to the sequence \( \{ \phi_\beta : \beta < \kappa^+ \} \). Second (Lemma 4.10), to get an \( m \times \mathcal{M} \) independent matrix immediately from a \( 1 \times \mathcal{M} \) independent matrix. Third (Section 5), to unify the notions of “good”, “mediocre”, and “OK” into one kind of point.

**Definition 4.2.** A step-family (over \( \kappa \), with respect to \( \widehat{\phi} \)) is an indexed collection of subsets of \( \kappa \), \( \{ E_s : s \in [\kappa]^{<\omega} \} \cup \{ A_\alpha : \alpha < \kappa^+ \} \), satisfying:
S1. $E_s \cap E_t = \emptyset$ for each distinct $s, t \in [\kappa]^{<\omega}$.
S2. $A_p = \bigcup \{ E_s : s \not\in p \}$ for each $p \in [\kappa^+]^{<\omega}$.
S3. $\hat{p} \subseteq s \Rightarrow |A_p \cap E_s| = \kappa$ for each $p \in [\kappa^+]^{<\omega}$ and $s \in [\kappa]^{<\omega}$.

Definition 4.3. If $\mathcal{F}$ is any filter on $\kappa$, then $\mathcal{F}^+ = \{ X \subseteq \kappa : \kappa \setminus X \notin \mathcal{F} \}$. 
$\mathcal{F}^\kappa = \mathcal{F}^\kappa(\kappa)$ is the Fréchet filter, $\{ X \subseteq \kappa : |\kappa \setminus X| < \kappa \}$.
So, $\mathcal{F}^\kappa = [\kappa]^{\kappa}$.

Definition 4.4. Given any index set $I$ and filter $\mathcal{F}$ on $\kappa$, the indexed collection $\{ E_i : s \in [\kappa]^{<\omega}, i \in I \} \cup \{ A^i : \alpha < \kappa^+, i \in I \}$ is an independent matrix of $|I|$ step-families (over $\kappa$) with respect to $\mathcal{F}$, $\sim$ iff:
1. For each fixed $i \in I$, $\{ E_i : s \in [\kappa]^{<\omega} \} \cup \{ A^i : \alpha < \kappa^+ \}$ is a step-family.
2. Given $n \in \omega$, $p_0, p_1, \ldots, p_{n-1} \in [\kappa^+]^{<\omega}$, $s_0, s_1, \ldots, s_{n-1} \in [\kappa]^{<\omega}$, and distinct $i_0, i_1, \ldots, i_{n-1} \in I$, if each $\hat{p}_i \subseteq s_i$ then
   $$\left( A^{i_0}_{p_0} \cap E^{s_0}_{i_0} \right) \cap \left( A^{i_1}_{p_1} \cap E^{s_1}_{i_1} \right) \cap \cdots \cap \left( A^{i_{n-1}}_{p_{n-1}} \cap E^{s_{n-1}}_{i_{n-1}} \right) \in \mathcal{F}^+.$$  (*)

Theorem 4.5. If $\kappa$ is regular and $\sim$ is any hat function, then there is an independent matrix of $2^\kappa$ step-families over $\kappa$ with respect to $\mathcal{F}^\kappa(\kappa)$, $\sim$.

The rest of this section is devoted to proving this theorem, which will get used in the construction in Section 6. So, $\sim$ will be fixed and the filter $\mathcal{F}$ will be understood to be $\mathcal{F}^\kappa$. We also assume that $\kappa$ is regular, in which case the following lemma is easily proved:

Lemma 4.6. Given $A_\xi \subseteq \kappa$ for $\xi < \kappa$, there are sets $B_\xi \subseteq \kappa$ such that:

$\Rightarrow A_\xi =^* B_\xi.$
$\Rightarrow A_\xi \subseteq^* A_\eta \Rightarrow B_\xi \subseteq B_\eta.$
$\Rightarrow A_\xi \cap A_\eta \Rightarrow B_\xi \cap B_\eta = \emptyset.$

Corollary 4.7 (($\kappa, \kappa$) separation property). Given subsets $S_\xi (\xi < \kappa)$ and $T_\eta (\eta < \kappa)$ of $\kappa$ such that each $S_\xi \perp T_\eta$, there is some $W \subseteq \kappa$ such that

$\forall \xi (W \supseteq S_\xi)$ and $\forall \eta (W \perp T_\eta)$.

Proof. By Lemma 4.6, we can assume that actually each $S_\xi \cap T_\eta = \emptyset$, and then we can set $W = \cup S_\xi$. [QED]

The following lemma does most of the work involved in getting a single step-family:

Lemma 4.8. Assume that $Q$ is a subset of $[\kappa^+]^{<\omega}$ which is closed downward and which contains $\emptyset$. Then there are $X_\alpha \subseteq \kappa$ for $\alpha < \kappa^+$ such that for all $p \in [\kappa^+]^{<\omega}$, $\left[ X_p \right] = \kappa$ if and only if $p \in Q$.

Proof. Let $(\kappa^+)^{<\omega}$ denote all the strictly increasing finite sequences from $\kappa^+$. Choose $Y_\sigma, Z_\sigma \in [\kappa]^\kappa$ (for $\sigma \in (\kappa^+)^{<\omega}$, max $\sigma < \alpha < \kappa^+$) so that

$\Rightarrow Z^\sigma_{\beta+1} \subseteq Y_{\sigma+1}$ whenever $\max \sigma < \beta < \kappa^+$
$\Rightarrow Z^\sigma_{\beta} \subseteq Y_{\sigma}$ whenever $\max \sigma < \alpha < \beta < \kappa^+$
$\Rightarrow Y_{\sigma+1} \cap Y_{\sigma} = Z^\sigma_{\alpha} \setminus Z^\sigma_{\alpha+1}.$
Then note that $Y_{\sigma - \alpha} \subseteq Y_{\sigma}$, and $\alpha \neq \beta \Rightarrow Y_{\sigma - \alpha} \perp Y_{\sigma - \beta}$. So, the $Y_{\sigma}$ form a tree of subsets of $\kappa$ with root $Y_{(1)} = \kappa$. Each node $Y_{\sigma}$ has $\kappa^+$ almost disjoint children, $Y_{\sigma - \alpha}$ ($\max \sigma < \alpha < \kappa^+$). The $Z_\beta$ let us achieve an instance of $(\kappa, \kappa^+)$ separation (see (5) below), when all we have is $(\kappa, \kappa)$ separation.

Now, we choose the sets $X_\beta$ by induction on $\beta$. Let $\Sigma_\beta = \{ \sigma \in (\kappa^+)^{<\omega} : \max \sigma = \beta \land \mathrm{ran} \sigma \in Q \}$. Using $(\kappa, \kappa)$-separation, choose $X_\beta$ so that:

1. $\tau \in \Sigma_\beta \Rightarrow X_\beta \supseteq \tau$
2. $(p \in [\beta]^\omega \land p \cup \{ \beta \} \not\subseteq Q) \Rightarrow X_\beta \perp X_p$
3. $\sigma \in (\beta)^{<\omega} \Rightarrow X_\beta \perp Z_{\beta + 1}^\sigma$
4. $(\sigma \in (\beta)^{<\omega} \land \sigma \supseteq \beta \not\subseteq \Sigma_\beta) \Rightarrow X_\beta \perp Z_\beta^\sigma$

Only (1) and (2) are needed to show that the $X_\alpha$ satisfy the lemma, but (3) and (4) are necessary in order to continue the induction. (3) implies:

5. $\beta \not\in \mathrm{ran} \tau \land \beta < \max \tau \Rightarrow X_\beta \perp Y_\tau$.

To see this, write $\tau = \sigma \cup \gamma \cup \rho$ where $\sigma \in (\beta)^{<\omega}$, $\gamma > \beta$, and $\rho$ fills out the rest of $\tau$. Then $Y_\tau \subseteq Y_{\sigma \cup \gamma} \subseteq Z_\sigma^\tau \subseteq Z_{\beta + 1}^\sigma$, so apply (3).

In order to see that $(\kappa, \kappa)$-separation really applies to get $X_\beta$ as above, we need to check that the $Y_{\tau}$ from (1) are almost disjoint from the $X_p$, $Z_{\beta + 1}^\sigma$, $Z_\beta^\sigma$ from (2,3,4). Fix $\tau = \pi \cup \beta \in \Sigma_\beta$, and note that by the definition of $\Sigma_\beta$, we have $\mathrm{ran}(\pi \cup \beta) \in Q$.

For (1) with (2), fix $p \in [\beta]^\omega$ such that $p \cup \{ \beta \} \not\subseteq Q$. Since $Q$ is closed downward, this implies that $\exists \alpha \in p$ such that $\alpha \not\in \tau$. By (5) applied inductively to $\alpha$, $X_\alpha \perp Y_{\tau}$ and therefore $X_p \perp Y_{\tau}$.

For (1) with (3) and (4), fix $\sigma \in (\beta)^{<\omega}$. Since $\tau = \pi \cup \beta$ cannot be an initial sequence of $\sigma$, there are three cases to consider.

Case 1: Neither of $\sigma$ and $\tau$ is an initial sequence of the other: Then $Z_{\beta + 1}^\sigma \subseteq Z_\beta^\sigma \subseteq Y_{\sigma} \perp Y_\tau$.

Case 2: $\sigma = \tau$: Then (4) cannot happen, and $Y_\tau = Y_{\sigma - \beta} \perp Z_{\beta + 1}^\sigma$.

Case 3: $\sigma$ is a proper initial sequence of $\tau$. Say $\sigma = \tau \cup \alpha \cup \rho$, where $\alpha < \beta$. Then $Y_\tau \subseteq Y_{\sigma - \alpha} \subseteq Z_\alpha^\sigma \perp Z_{\beta + 1}^\sigma$.

We now prove: $|X_p| = \kappa \iff q \in Q$. Fix $q \in [\kappa^+]^{<\omega}$. If $q \not\subseteq Q$ then write $q = p \cup \{ \beta \}$ where $\beta$ is the maximum element of $q$ and $p \in [\beta]^{< \omega}$; then (2) implies $|X_p| = |X_\beta \cap X_p| < \kappa$. If $q \subseteq Q$, write $q = \{ \alpha_0, \alpha_1, \ldots, \alpha_n \}$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Let $\sigma = (\alpha_0, \alpha_1, \ldots, \alpha_n)$. Then each $(\alpha_0, \alpha_1, \ldots, \alpha_i) \in \Sigma_{\alpha_i}$, so $Y_{\sigma} \subseteq Y_{(\alpha_0, \ldots, \alpha_i)} \subseteq X_{\alpha_i}$ by (1). Therefore $Y_{\sigma} \subseteq X_{\alpha_i}$ so $|X_p| = \kappa$.

Lemma 4.9. Given any hat function $\hat{\tau}$, there exists a step-family, with respect to $\hat{\tau}$, $\{ E_s : s \in [\kappa]^{< \omega} \} \cup \{ A_\alpha : \alpha < \kappa^+ \}$.

Proof. Let $\bigcup \{ E_s : s \in [\kappa]^{< \omega} \} = \kappa$ be any partition of $\kappa$ into sets of size $\kappa$. For $\beta < \kappa^+$, we define $A_\beta$ roughly by defining each $A_\beta \cap E_s$.

For $s \in [\kappa]^{< \omega}$, set $Q_s = \{ \hat{p} \in [\kappa^+]^{< \omega} : \hat{p} \subseteq s \}$. Then $Q_s$ is as in Lemma 4.8, so we can choose subsets $X_{\alpha}^s$ of $E_s$ such that $|X_{\alpha}^s| = \kappa \iff \hat{p} \subseteq s$. 

As a first approximation to the \( A_\beta \), define \( C_\beta = \bigcup \{ X_\beta^s : s \in [\kappa]^{<\omega} \} \). Then \( C_\beta \cap E_s = X_\beta^s \) so \( |C_\beta^p \cap E_s| = \kappa \iff \widehat{p} \subseteq s \). These \( C_\beta \) satisfy part (S3) of Definition 4.2, but they may fail to satisfy (S2). We have \( |C_\beta^p \cap E_s| < \kappa \)
whenever \( s \not\supseteq \widehat{p} \), but we need \( |C_\beta^p \cap \bigcup \{ E_s : s \not\supseteq \widehat{p} \}| < \kappa \).

Now, choose \( A_\beta \) (by induction on \( \beta < \kappa^+ \)) such that:
(i) \( A_\beta \supseteq C_\beta \cap E_s \) for all \( s \in [\kappa]^{<\omega} \).
(ii) \( A_\beta \perp E_s \cap C_\beta \) for all \( s \in [\kappa]^{<\omega} \).
(iii) \( A_\beta \perp \bigcup \{ A_\beta^p \cap E_t : t \not\supseteq p \cup \{ \beta \} \} \) for all \( p \in [\beta]^{<\omega} \).

Assuming this can be done, (i) and (ii) give us also:
(iv) \( A_\beta^p \cap E_s =^* C_\beta^p \cap E_s \).

By (iv), the \( A_\beta \) also satisfy (S3), and (iii) implies that the \( A_\beta \) satisfy (S2).

To prove that such \( A_\beta \) can be chosen, we apply (\( \kappa, \kappa \)) separation, but (i,iii) requires us to verify that for all \( s \in [\kappa]^{<\omega} \) and all \( p \in [\beta]^{<\omega} \):

\[
D_{\beta,s} := (C_\beta \cap E_s) \cap \bigcap \{ A_\beta^p \cap E_t : t \not\supseteq p \cup \{ \beta \} \} =^* \emptyset.
\]

If \( s \supseteq p \cup \{ \beta \} \), then \( D_{\beta,s} = \emptyset \), so assume that \( s \not\supseteq p \cup \{ \beta \} \). Then \( D_{\beta,s} = C_\beta \cap E_s \cap A_\beta^p \). Applying (iv) inductively, \( D_{\beta,s} =^* C_\beta^p \cap E_s \), which has size less than \( \kappa \).

**Lemma 4.10.** If \( \kappa \) is regular, \( \sim \) is any hat function, and \( n < \omega \), then there is an independent matrix of \( n \) step-families over \( \kappa \) with respect to \( \mathcal{F} \mathcal{R}(\kappa) \), \( \sim : \{ E_i^s : s \in [\kappa]^{<\omega}, i < n \} \cup \{ A_\alpha^i : \alpha < \kappa^+, i < n \} \).

**Proof.** We apply Lemma 4.9, changing the index set to \( [n \times \kappa^+]^{<\omega} \). For \( P \in [n \times \kappa^+]^{<\omega} \) and \( i < n \), let \( P_i = \{ \alpha : (i, \alpha) \in P \} \), and define \( \widehat{P} = ((\{ 0 \} \times \widehat{P}_0) \cup (\{ 1 \} \times \widehat{P}_1) \cup \cdots \cup (\{ n-1 \} \times \widehat{P}_{n-1}) \).

Then \( \sim \) is a hat function from \( [n \times \kappa^+]^{<\omega} \) to \( [n \times \kappa]^{<\omega} \). So, we have a step-family with respect to \( \sim \), \( \{ E_i^s : S \in [n \times \kappa]^{<\omega} \} \cup \{ A_{i,\alpha}^s : i < n, \alpha < \kappa^+ \} \).

For each \( i < n \) and \( s \in [\kappa]^{<\omega} \), let \( E_i^s = \bigcup \{ E_S : S_i = s \} \). Since all the \( E_S \) are disjoint, the \( E_i^s \) for each fixed \( i \), are also disjoint. Let \( A_{i,\alpha}^s = A_{i,\alpha}^{s_i} \).

Fix \( i < n \) and \( p \in [\kappa^+]^{<\omega} \); we verify condition (S2) in the definition of step-family; that is, \( A_{i,\alpha}^s \perp \bigcup \{ E_i^s : s \not\supseteq \widehat{p} \} \). Now \( A_{i,\alpha}^s = \bigcap (\{ i \} \times p \) and

\[
\bigcup \{ E_i^s : s \not\supseteq \widehat{p} \} = \bigcup \{ E_S : S \not\supseteq \{ i \} \times p \},
\]
so we apply (S2) with respect to \( \sim \).

Observe that for \( S \in [n \times \kappa]^{<\omega} \) and \( P \in [n \times \kappa^+]^{<\omega} \), we have

\[
E_S = E_S^0 \cap E_S^1 \cap \cdots \cap E_S^{n-1}
A_P = A_P^0 \cap A_P^1 \cap \cdots \cap A_P^{n-1}.
\]

To prove independence (which implies (S3) for each step-family), fix \( p_{i, \ell} \) for \( \ell < n \) as in Definition 4.4.2 (here, \( I = n \) and \( i_{\ell} = \ell \)). Assume each
\[ \hat{p}_i \subseteq s_i. \text{ Define } P, S \text{ so that } P_i = p_i \text{ and } S_i = s_i \text{ for each } i. \text{ Then } P \subseteq S, \text{ so } |A_{\hat{p}} \cap E_S| = \kappa, \text{ which implies } (*) \text{ in Definition 4.4.} \]

**Proof of Theorem 4.5.** We apply Theorem 3.4 (or Theorem 3.6 in the case \( \kappa = \omega \)), but one must use a little care because the definition of step-family involves infinite unions of the \( E_i \). We handle this by adding a “name”, \( b_t \), to represent \( B_t := \bigcup\{E_s : s \not\supseteq t\} \).

Let \( B \) be the free algebra generated by elements \( a_\alpha \) for \( \alpha < \kappa^+ \), together with elements \( e_s \) and \( b_t \) for \( s \in [\kappa]^{<\omega} \). Let \( J \) be generated by:

1. \( e_s \wedge e_t \) whenever \( s, t \) are distinct.
2. \( e_s \setminus b_t \) whenever \( s \not\supseteq t \).
3. \((\bigwedge_{\alpha \in p} a_\alpha \wedge b_p)\) for each \( p \in [\kappa^+]^{<\omega} \).

Let \( P \) contain:

1. \( e_s \wedge \bigwedge_{\alpha \in p} a_\alpha \) whenever \( \hat{p} \subseteq s \).

By Lemma 4.10, there is an \( m \times M \) independent matrix for each finite \( m \).
Hence, there is a \( 2^m \times M \) independent matrix \( \langle h_\alpha : \alpha < 2^m \rangle \) in \( P(\kappa)/\kappa \).

Now, fix \( \alpha \). Since \([\kappa]^{<\omega} = \kappa\), which is regular, we can apply Lemma 4.6 to choose representatives \( E^\alpha_s \) and \( B^\alpha_s \) such that \( h_\alpha(e_s) = [E^\alpha_s] \), \( h_\alpha(b_s) = [B^\alpha_s] \), the \( E^\alpha_s \) are really disjoint (using \( J(1) \)), and \( E^\alpha_s \subseteq B^\alpha_t \) whenever \( s \not\supseteq t \) (using \( J(2) \)). Then condition \( J(3) \) ensures that \( A_{\hat{p}} \perp \bigcup\{E_s : s \not\supseteq \hat{p}\} \). Condition \( J(1) \) now ensures that \( \hat{p} \subseteq s \Rightarrow |A_{\hat{p}} \cap E_s| = \kappa \).

Thus, the matrix gives us \( 2^\kappa \) independent step-families in the sense of Definition 4.4. \( \square \)

### 5. Hatpoints

Using the hat functions, we see that “good” and “mediocre” are different flavors of the same notion. Throughout, \( \kappa \) is a fixed regular cardinal.

**Definition 5.1.** Let \( \sim : [\kappa^+]^{<\omega} \rightarrow [\kappa]^{<\omega} \) be a hat function and let \( x \) be a point in \( X \). Then \( x \) is a \( \sim \)-point in \( X \) iff, given neighborhoods \( U_r \ (r \in [\kappa]^{<\omega}) \) of \( x \), there are neighborhoods \( V_\alpha \ (\alpha < \kappa^+) \) of \( x \) such that \( V_\alpha \subseteq U_r \) for each non-empty \( r \in [\kappa^+]^{<\omega} \).

For example, if \( \hat{p} = p \cap \kappa \) for all \( p \), then \( x \) is a \( \sim \)-point iff \( x \) is \( \kappa^+ \)-good. In this case, the \( V_\alpha \) for \( \alpha \geq \kappa \) are irrelevant.

**Lemma 5.2.** There is a hat function \( \sim : [\kappa^+]^{<\omega} \rightarrow [\kappa]^{<\omega} \) such that every \( \sim \)-point is both \( \kappa^+ \)-mediocre and \( \kappa^+ \)-good.

**Proof.** Let \( \langle \varphi_\beta : \beta < \kappa^+ \rangle \) be as in Definition 2.3, but assume also that \( \varphi_\beta(\alpha) = \alpha \) whenever \( \alpha < \beta \leq \kappa \). Let \( \sim \) be the derived hat function and let \( x \) be a \( \sim \)-point. Applying Definition 5.1 with \( r = \{\alpha, \beta\} \) shows that \( x \) is \( \kappa^+ \)-mediocre. To show that \( x \) is \( \kappa^+ \)-good, let \( W_\alpha = V_\alpha \cap V_\kappa \). Then for \( r \in [\kappa]^{<\omega} \), \( r \cup \{\kappa\} = r \), so \( W_\alpha = V_{r \cup \{\kappa\}} \subseteq U_r \). \( \square \)

**Remark 5.3.** In the definition (5.1) of “hatpoint”, it is sufficient to consider only monotone sequences of neighborhoods, \( \langle U_r : r \in [\kappa]^{<\omega} \rangle \) of \( x \); that is \( r \subseteq s \Rightarrow U_r \supseteq U_s \).
Proof. Replace each $U_r$ by $\cap \{ U_s : s \subseteq r \}$.

Keisler’s original definition of “good” had “monotone”, and restricting to monotone sequences will simplify the construction in Section 6.

In the definition of hat function and hatpoint, one could more generally consider $\sim : [\theta]^{<\omega} \to [\kappa]^{<\omega}$. For example, fix any $\theta$, let $\kappa = \omega$, and define $\hat{r} = \{ 0, 1, \ldots, |r| \}$. Then the $\sim$ points are just the $\theta$-OK points from [5].

6. Proof of Theorem 1.2

Applying Lemmas 5.2, 2.6 and 2.5, Theorem 1.2 is immediate from the following more general result:

**Theorem 6.1.** If $\kappa$ is regular and $\sim : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ is a hat function, then there is an $x \in u(\kappa)$ which is a $\sim$ point in $u(\kappa)$.

**Proof.** Apply Theorem 4.5, and fix a matrix of $2^\kappa$ step-families (over $\kappa$), 
$\{ E^i_s: s \in [\kappa]^{<\omega}, i \in 2^\kappa \} \cup \{ A^i_\alpha: \alpha < \kappa^+, i \in 2^\kappa \}$, which is independent with respect to $\mathcal{F}\mathcal{R}(\kappa)$. Assume that for each $\alpha$, we have, in addition to (S1), (S2), (S3) in Definition 4.2:

S4. $\bigcup \{ E^i_s: s \in [\kappa]^{<\omega} \} = \kappa$.
S5. $A^i_\alpha \subseteq \bigcup \{ E^i_s: s \supseteq \hat{p} \}$ for each $p \in [\kappa^+]^{<\omega}$.

To get (S4), first shrink the $A^i_\alpha$ so that each $A^i_\alpha \subseteq \bigcup \{ E^i_s: s \in [\kappa]^{<\omega} \}$, and then expand the $E^i_s$ so that $\{ E^i_s: s \in [\kappa]^{<\omega} \}$ forms a partition of $\kappa$. Then, (S5) follows from (S2, S4).

We plan to construct the ultrafilter $x$ by induction, in $2^\kappa$ steps. We have two basic tasks: making $x$ “ultra” and making $x$ a $\sim$ point. To accomplish these, let $B_\mu, C^\mu_\rho$ be subsets of $\kappa$, for $\mu < 2^\kappa$, $r \in [\kappa]^{<\omega}$, so that:

1. $\mathcal{P}(\kappa) = \{ B_\mu: \mu < 2^\kappa \land \mu \equiv 0 \mod 2 \}$.
2. Each $\langle C^\mu_\rho: r \in [\kappa]^{<\omega} \rangle$ is monotone (see Remark 5.3), and every monotone sequence in $\mathcal{P}(\kappa)[[\kappa]^{<\omega}]$ appears as $\langle C^\mu_\rho: r \in [\kappa]^{<\omega} \rangle$ for $2^\kappa$ distinct $\mu$ such that $\mu \equiv 1 \mod 2$.

We shall construct $x$ as an increasing union of filters: $x = \bigcup_{\mu<2^\kappa} F_\mu$. Set $\mathcal{F}_0 = \mathcal{F}\mathcal{R}(\kappa)$ and $I_0 = 2^\kappa$. We define $F_\mu$ and $I_\mu (\mu < 2^\kappa)$ so that:

1. If $\mu < \nu$ then $F_\mu \subseteq F_\nu$ and $I_\mu \supseteq I_\nu$.
2. For limit $\nu: \mathcal{F}_\nu = \bigcup_{\mu<\nu} F_\mu$ and $I_\nu = \bigcap_{\mu<\nu} I_\mu$.
3. Each $I_\mu \setminus I_{\mu+1}$ is finite.
4. Each $F_\mu$ is a filter on $\kappa$, and the matrix $\{ E^i_s: s \in [\kappa]^{<\omega}, i \in I_\mu \} \cup \{ A^i_\alpha: \alpha < \kappa^+, i \in I_\mu \}$ of remaining step-families is independent w.r.t. $F_\mu$.
5. If $\mu \equiv 0$ then either $B_\mu \in F_{\mu+1}$ or $\kappa \setminus B_\mu \in F_{\mu+1}$.
6. If $\mu \equiv 1$ and each $C^\mu_\rho \in F_\mu$, then there are $D^\mu_\alpha \in F_{\mu+1}$ for $\alpha < \kappa^+$ such that $\forall p \in [\kappa^+]^{<\omega}$ $\left[ D^\mu_\alpha \subseteq \bigcup \{ C^\mu_\rho \} \right]$.

As usual in these constructions, there is no problem at limits, so we proceed to describe the successor step. For $E \subseteq \mathcal{P}(\kappa)$, let $\langle E \rangle$ denote the filter generated by $E$. 
If $\mu \equiv 0$: If $\langle \mathcal{F}_\mu \cup \{B_\mu\} \rangle$ is a proper filter and the matrix of step-families
\[
\{E_i : s \in [\kappa]^{<\omega}, \ i \in I_\mu \} \cup \{A^i_\alpha : \alpha < \kappa^+, \ i \in I_\mu \}
\]
is independent w.r.t. $\langle \mathcal{F}_\mu \cup \{B_\mu\} \rangle$, then $\mathcal{F}_{\mu+1} = \langle \mathcal{F}_\mu \cup \{B_\mu\} \rangle$ and $I_{\mu+1} = I_\mu$. Otherwise, fix
\[n \in \omega, \ \text{distinct} \ i_l \in I_\mu (l < n), \ \text{and} \ \hat{\rho} \subseteq s_l (l < n), \ \text{such that:}
\]
\[
B_\mu \cap \left( A^{i_0}_{\mathcal{F}_\mu} \cap E_{\mathcal{F}_\mu}^{i_0} \right) \cap \left( A^{i_1}_{\mathcal{F}_\mu} \cap E_{\mathcal{F}_\mu}^{i_1} \right) \cap \cdots \cap \left( A^{i_{n-1}}_{\mathcal{F}_\mu} \cap E_{\mathcal{F}_\mu}^{i_{n-1}} \right) \notin \mathcal{F}_\mu.
\]
Set $\mathcal{F}_{\mu+1} = \langle \mathcal{F}_\mu \cup \left\{ A^{i_0}_{\mathcal{F}_\mu}, \ldots, A^{i_{n-1}}_{\mathcal{F}_\mu}, \ E^{i_0}_{\mathcal{F}_\mu}, \ldots, E^{i_{n-1}}_{s_{n-1}} \right\} \rangle$ and set $I_{\mu+1} = I_\mu \setminus \{i_0, i_1, \ldots, i_{n-1}\}$. Then $\kappa \setminus B_\mu \in \mathcal{F}_{\mu+1}$, and we leave it to the reader to check that condition (4) holds for $\mathcal{F}_{\mu+1}$ and $I_{\mu+1}$.

If $\mu \equiv 1$: Assume each $C^i_s (s \in [\kappa]^{<\omega})$ is in $\mathcal{F}_\mu$. (Otherwise, set $\mathcal{F}_{\mu+1} = \mathcal{F}_\mu$ and $I_{\mu+1} = I_\mu$.) Choose $i \in I_\mu$ and set $I_{\mu+1} = I_\mu \setminus \{i\}$. We shall leave off the superscripts $\mu$ and $i$ when they are clear from context. For $\alpha < \kappa^+$, define
\[
D_\alpha = A_\alpha \cap \bigcup \left\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \right\}.
\]
Then for $p \in [\kappa^+]^{<\omega}$,
\[
D_p = A_p \cap \bigcup \left\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \right\}
\]
\[
\subseteq \bigcup \left\{ E_s : s \subseteq \hat{\rho} \right\} \cap \bigcup \left\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \right\} = \bigcup \left\{ C_s \cap E_s : s \subseteq \hat{\rho} \right\} \subseteq \bigcup \left\{ C_s : s \supseteq \hat{\rho} \right\} = C_{\hat{\rho}}
\]
by monotonicity of $\langle C_p : p \in [\kappa]^{<\omega} \rangle$. Set $\mathcal{F}_{\mu+1} = \langle \mathcal{F}_\mu \cup \{D_\alpha : \alpha < \kappa^+\} \rangle$. To verify condition (4): Any element of $\mathcal{F}_{\mu+1}$ is of the form $B \cap D_p$ for some $B \in \mathcal{F}_\mu$ and $p \in [\kappa^+]^{<\omega}$. But $D_p \supseteq A_p \cap C_{\hat{\rho}} \cap E_{\hat{\rho}}$, and $C_{\hat{\rho}} \in \mathcal{F}_\mu$, so condition (4) for $\mathcal{F}_{\mu+1}$ now follows easily from condition (4) for $\mathcal{F}_\mu$. \hfill $\square$

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