Logic PhD Qualifying Exam

February 1971

Majors & Minors:

Do five problems, no more than three from a single section.

Third Area:

Do any four problems.

A. Model Theory

1. Let D be an ultrafilter over I. For each $i \in I$ let $\mathfrak{A}_i$ be a model and $\langle X_i, <_i \rangle$ a set of indiscernibles in $\mathfrak{A}_i$. Prove that the ultraproduct $\Pi_D \langle X_i, <_i \rangle$ is a set of indiscernibles in $\Pi_D \mathfrak{A}_i$.

2. A model $\mathfrak{A}$ is said to omit a set of formulas $\Sigma(x) = \{\sigma_0(x), \sigma_1(x), \ldots\}$ if $\mathfrak{A} \models \neg \exists x (\sigma_0(x) \wedge \sigma_1(x) \wedge \sigma_2(x) \wedge \ldots)$. Assume

   (i) T has a model omitting $\Sigma(x)$

   (ii) Every model of T which omits $\Sigma(x)$ has a proper elementary extension omitting $\Sigma(x)$.

   Prove that T has models of arbitrarily large power which omit $\Sigma(x)$.

3. Let S be a set of elementarily equivalent models. Show that there is a model $\mathfrak{A}$ such that every $\mathfrak{B} \in S$ is elementarily embeddable in $\mathfrak{A}$.

4. Let L be a language with a unary predicate P_X for each set $X \subset \omega$. Let $\mathfrak{A}$ have base set ω and let each P_X be interpreted by X in $\mathfrak{A}$. For each $\mathfrak{B} \succ \mathfrak{A}$ and $b \in B$, let

   $U_b = \{X \subseteq \omega : \mathfrak{B} \models P_X(b)\}$. Prove that U_b is an ultrafilter over ω.

B. Set Theory

Notation: V_α denotes the set of all sets of rank $< \alpha$, that is $V_0 = 0$, $V_\alpha = \bigcup_{\beta<\alpha} S(V_\beta)$. Let ZF Zermelo Fraenkel set theory (without choice). ZFC = ZF + choice.

5. Show in ZF that for every x there is an ordinal α which cannot be mapped one-one into x.

6. What, if anything, is wrong with the following argument?

Let ZFC_n be the first n axioms of ZFC.

a) By the reflection principle, for all $n < \omega$,

$ZFC \vdash (\exists \alpha)(V_\alpha$ is a model of $ZFC_n)$.

b) For all $n < \omega$, $ZFC \vdash (ZFC_n$ has a model$)$.

c) $ZFC \vdash$ (If every finite subset of ZFC has a model, then ZFC has a model).

d) $ZFC \vdash$ (ZFC has a model).

e) $ZFC \vdash$ (ZFC is consistent).

f) by Gödel's incompleteness theorem, ZFC is inconsistent.

7. Prove that

$$\aleph_\omega < \aleph_\omega^{\aleph_0}.$$

8. Show in ZF that every countable transitive set belongs to V_{ω_1}.

C. Recursion Theory.

9. Let T be a set of sentences in first order logic whose set of Gödel numbers is r.e. Show that there is a set T' of sentences with a recursive set of Gödel numbers such that for each sentence φ,

$$T \vdash \varphi \quad \text{if and only if} \quad T' \vdash \varphi.$$

10. Show that the set of Gödel numbers of true sentences of arithmetic is hyperarithmetical.

11. Let φ_e be the recursive partial function of one variable with Gödel number e. Show that there is an e such that φ_e is total and its range has cardinality e.

12. Show that if $P(x, y)$ is a recursively enumerable predicate then there is a recursive partial function ψ such that:

a) domain $(\psi) = \{x : \exists y\, P(x, y)\}$

b) If $\psi(x)$ is defined then $P(x, \psi(x))$.

D. Admissible sets (special request).

13. Let κ be an uncountable cardinal and let $A_\kappa = \{b : b$ is definable in L_κ by a Σ_1 formula without parameters$\}$.

Prove that A_κ is admissible.

14. Assume there is an ω-model of ZF. Let A be an admissible set such that $\alpha \in A$ for some admissible ordinal $\alpha > \omega$. Show that $\langle A, \in \rangle \models$ (there is an ω-model of ZF).