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A. Elementary Problems.

1. Let T be a finitely axiomatizable theory with only a countable number of complete extensions. Prove that one of these complete extensions is finitely axiomatizable.

2. Let T be a model-complete theory. Prove that T is complete if and only if for every two models $\mathfrak{A}$, $\mathfrak{B}$ of T there is a model $\mathfrak{D}$ of T such that both $\mathfrak{A}$, $\mathfrak{B}$ are isomorphic to submodels of $\mathfrak{D}$.

3. Prove in ZFC that if $\omega \leq \lambda \leq \kappa$ then

$$(\kappa^+)^\lambda = \max(\kappa^\lambda, \kappa^+) .$$

4. Prove that for every set X and ordinal α there is a function f with domain α such that $f(0) = X$ and for all $\beta < \gamma < \alpha$, $f(\beta)_{f(\beta)} \subseteq f(\gamma)$.

5. Let $\mathfrak{A} = \langle A, < \rangle$, $\mathfrak{B} = \langle B, <' \rangle$ be infinite linear orderings. Prove that $\mathfrak{B}$ is isomorphic to a submodel of some elementary extension of $\mathfrak{A}$.

Qualifying Examination in Logic

8. Model Theory

Always assume the language is countable.

1. Let T be a complete theory which has $2^{\aleph_0}$ complete types. Show that for each infinite cardinal $\kappa \le 2^{\aleph_0}$ there are models of T of arbitrarily large cardinality which realize exactly κ complete types.

2. Let κ be an inaccessible cardinal and let $\mathfrak{A}$ be a saturated model of power κ. Prove that $\mathfrak{A}$ is the union of a proper elementary chain $\mathfrak{A}_\beta$, $\beta < \kappa$, of models isomorphic to $\mathfrak{A}$.

3. Let D be an ultrafilter and let $\mathfrak{A} \times \mathfrak{B}$ be the direct product of $\mathfrak{A}$ and $\mathfrak{B}$. Prove that $\Pi_D(\mathfrak{A} \times \mathfrak{B}) \cong \Pi_D\mathfrak{A} \times \Pi_D\mathfrak{B}$.

4. Prove that the complete theory of $\langle Z, < \rangle$ has exactly 2 non-isomorphic countably homogeneous models.

C. Recursion Theory.

1. Let $\prec$ be a recursive well ordering of type α. Show that there is a recursive well ordering of type ω^{α} (ordinal exponentiation).

2. Let d be a Turing degree with the property that

$$\exists x \in d \; \forall y \leq_T d [y \leq_m x]$$

direct diagonalization

Prove that $O' \not\leq d$.

3. Prove that there is a recursive linear ordering $\prec$ such that $\prec$ is not a well ordering but such that $\prec$ has no arithmetic descending sequences.

4. Suppose that for every set A of natural numbers there is a (unique) set B such that:

$$\forall n [K_B(n) = \varphi_e^{K_A}(n)] \quad .$$

Write $\mathfrak{J}(A) = B$ if the above holds. Prove that there is a recursive function f and a recursive relation R such that for all A,

$$\mathfrak{J}(A) = \{n \mid R(\overline{K}_A(f(x)))\} \quad .$$

D. Set Theory

1. Prove that if $\langle R(\alpha), \in \rangle$ is a model of Z^*, then α is a cardinal. Show that if there is such an α then the least such α has cofinality ω.

2. Assume that ZFC has a standard model. Let T_n be the set of all sentences φ such that in every standard model M of ZFC,

$$M \models (L_{\omega_n} \text{ is a model of } \varphi) .$$

Prove that $T_1 \subseteq T_2 \subseteq \ldots \subseteq T_n \subseteq \ldots$.

3. For $x, y \subseteq \omega$ define $x \leq_L y$ iff $x \in L[y]$. Assume that $\forall x, y[x \leq_L y$ or $y \leq_L x]$. Prove that $2^{\aleph_0} \leq \aleph_2$.

4. Let κ be inaccessible. Let

$$A = \{\alpha < \kappa \mid R(\alpha) \models ZFC\} .$$

Prove that A is not closed but that it contains a set B which is closed and unbounded in κ.