Aug 75

A. Elementary Problems

1) a) Prove in ZFC that there is no sequence of sets, $\langle x_n : n \in \omega \rangle$, such that $x_{n+1} \in x_n$ for all n.

b) Show that if ZFC is consistent, it has a model $\langle A, E \rangle$ such that for some $x_n \in A$ $(n \in \omega)$, $x_{n+1} E x_n$ for all n.

c) How do you reconcile (a) and (b)?

2) For any cardinal κ, let $H(\kappa) = \{x \mid |TC(x)| < \kappa\}$

a) Show, in ZF, that $H(\kappa)$ is a set.

b) Show, in ZFC, that for $\kappa \geq \omega$, κ is regular iff $H(\kappa) = \{x ; |x| < \kappa \ \& \ x \subsetneq H(\kappa)\}$.

3) Let T be the theory of models (A, f) such that f is a permutation of A with no finite cycles. Show that T is not finitely axiomatizable.

4) Show that there is a model $\mathfrak{A} \models P$ such that for some $n \in A$, n is infinitely large and there is no smaller $m \in A$ realizing the same type as n.

5) Let T be the theory of models $\langle A, U \rangle$ such that U and $A-U$ are both infinite (U is unary). Show that T is model-complete.

B. Model Theory

1) Let λ and κ be infinite cardinals with $\kappa \leq \lambda$ and κ regular. Show that there is an $\mathfrak{A} \models P$ such that $\mathfrak{A}$ has cardinality λ and cofinality κ.

2) Let $\mathfrak{A}$ and $\mathfrak{B}$ be ω-saturated structures for a countable language. Show $\mathfrak{A} \times \mathfrak{B}$ is ω-saturated.

3) Let T be a consistent theory in a countable language. Assume every model of T of cardinality ω_2 is ω_1-saturated. Show T is ω_1-categorical. (Hint: first show T is ω-stable).

4) Let T be a complete theory in a countable language and let $\sum(x)$ be a non-atomic type over T. Show that T has a model M which realizes $\sum(x)$ but is not ω-saturated.

5. Let $\mathfrak{A}$ be a countable infinite arithmetically saturated model. Show that $\mathfrak{A}$ has a non-trivial automorphism.

C. Recursion Theory

1) Show that there are distinct $a, b, c \in \omega$ such that $\varphi_a(b) = c$, $\varphi_b(c) = a$, and $\varphi_c(a) = b$.

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2) Show that there is an infinite r. e. set with no infinite recursive subsets.

3) a) Show that there is a recursive $S \subset \omega$ such that $\{\varphi_e : e \in S\}$ = the set of primitive recursive functions of one variable.

b) Show that the characteristic function of such an S is not primitive recursive.

4) Let $A, B \subseteq \omega$, $A \cap B = 0$, and A, B both Π^0_3. Show that there is a Δ^0_3 C with $A \subseteq C$ and $B \cap C = 0$.

D. Set Theory

1) Assume $\alpha < \omega_1$ and there exists a transitive $M \models$ ZFC with $M \cap ON = \alpha$. Show there are transitive $N_1, N_2 \models$ ZFC with $N_1 \cap ON = N_2 \cap ON = \alpha$, $N_1 \equiv N_2$, and $N_1 \not\cong N_2$.

2) a) Assume:

$X_\alpha \subseteq \omega_1$ $(\alpha < \omega_1)$, $\forall \alpha < \beta$ $(X_\beta \subseteq X_\alpha)$, and (*)

$Y = \{\alpha : \forall \beta < \alpha \ (\alpha \in X_\beta)\}$.

Show that if each X_α is c.u.b., then Y is c.u.b.

b) Find X_α $(\alpha < \omega_1)$ satisfying (*) with each X_α stationary and Y not stationary.

3) Let α be a limit ordinal. Show α is a regular cardinal iff: whenever $\circ : \alpha \times \alpha \to \alpha$ is such that $(\alpha, \circ)$ is a group, $\exists \gamma < \alpha$ (γ is a subgroup of α).

4) Assume $V = L$ and $\exists$ an inaccessible cardinal. Find a theory $T \supseteq$ ZF in the language of set theory such that:

a) T has no uncountable transitive model but

b) $\forall \alpha < \omega_1 \exists M$ ($M \models T$ & M transitive & $\alpha \in M$).

ZF + $\exists x \subseteq \omega$ $x \notin L$