Qualifying Examination in Logic

September 1978

<u>INSTRUCTIONS</u>: Do five problems, at most two from part A.

Do not use the continuum hypothesis.

<u>Glossary</u>:

$|x|$ = cardinality of x.

$cf(\lambda)$ = cofinality of λ.

$R(\alpha) = \{x : x \text{ has rank} < \alpha\}$.

$\mathfrak{A}$ is ω-homogeneous if whenever $(\mathfrak{A}, a_1, \cdots, a_n) \equiv (\mathfrak{A}, b_1, \cdots, b_n)$,

we have $\forall c \exists d \ (\mathfrak{A}, a_1, \cdots, a_n, c) \equiv (\mathfrak{A}, b_1, \cdots, b_n, d)$.

$\mathbb{Q}$ = set of rational numbers.

$\Diamond(S)$ means that $S \subseteq \omega_1$ and there is a family of sets $A_\alpha \subseteq \alpha$, $\alpha \in S$, such that for all $A \subseteq \omega_1$,

$$\{\alpha \in S : A \cap \alpha = A_\alpha\}$$

is stationary in ω_1.

$\Diamond$ means $\Diamond(\omega_1)$.

ZF is Zermelo-Fraenkel set theory.

A. Elementary Problems

A1. Let $\mathbb{C} = \langle C, +, \cdot, 0, 1\rangle$ be the field of complex numbers and let $R \subseteq C$ be the set of real numbers. Show that R is not definable in $\mathbb{C}$.

A2. Let $\mathbb{H}\mathbb{F} = \langle HF, \in \rangle$ be the structure of hereditarily finite sets. State and prove a version of Tarski's Theorem on Truth which applies to $\mathbb{H}\mathbb{F}$.

A3. Let T be a universal theory. Assume $T \models \forall x \exists y\, P(x, y)$. Show that there are terms $t_1(x), \cdots, t_n(x)$ such that $T \models \forall x \bigvee_{m=1}^{n} P(x, t_m(x))$.

A4. Let κ be a cardinal and let

$$\lambda = \sup \{2^{\alpha} : \alpha < \kappa\} .$$

Show that $cf(\lambda) = cf(\kappa)$ or $cf(\lambda) > \kappa$.

A5. Find the mistake in the following proof.

a) For each finite $S \subseteq ZF$, $ZF \vdash (\exists \alpha)(\langle R(\alpha), \in \rangle$ is a model of $S)$.

b) $ZF \vdash$ (If every finite $S \subseteq ZF$ has a model, then ZF has a model).

c) By a) and b) $ZF \vdash$ (ZF has a model).

d) By Gödel's second theorem and c), ZF is inconsistent.

B. Model Theory

B1. Let T be the theory with the axioms

$$\begin{cases} \forall y \, \exists x \; y = F(x) \\ \forall x \, \forall y \; F(x) = F(y) \rightarrow x = y \end{cases}$$

Show that every complete extension of T is ω-stable and has Morley rank at most two.

B2. Let $\mathfrak{A} = \langle A, <, \cdots \rangle$ be an ω-homogeneous model for a countable language such that $<$ well orders A. Prove that A has cardinality at most 2^ω.

In problems B3-B5, let T be a countable complete theory whose models are infinite.

B3. Prove that T has a family of countable models $\mathfrak{A}_S$, $S \subset \omega$, such that if S is a proper subset of T then $\mathfrak{A}_S$ is a proper elementary submodel of $\mathfrak{A}_T$. Hint: Use indiscernibles.

B4. Show that T has an ω-homogeneous model of power ω_1 with only countably many types. Hint: Similar to Vaught's two-cardinal argument.

B5. (Shelah). Let S be a set of fewer than 2^ω types $\Gamma(x)$ which are maximal consistent with T and locally omitted by T. Prove that T has a model which simultaneously omits each $\Gamma(x) \in S$.
Hint: Represent the Henkin construction by a binary tree.

C. Recursion Theory

C1. Find a function $d : \omega \times \omega \rightarrow \mathbb{Q}$ such that:

a) d is recursive.

b) d is a metric.

c) the set $\{n \in \omega : n$ is isolated in the space $(\omega, d)\}$ is not recursive.

C2. Show that there is a set of Turing degrees $\{d_q : q \in \mathbb{Q}\}$ such that $q < r$ implies

$$0 < d_q < d_r < 0' \quad .$$

C3. Let $f : \omega \rightarrow \omega$ be recursive. Show that there is a function $g : \omega \rightarrow \omega$ such that f is primitive recursive in g and g has a primitive recursive graph.

C4. Let $C(X)$ be a Σ^1_1 predicate with no Δ^1_1 solutions. Prove that the set of solutions of $C(X)$ has cardinality 2^ω .

The following problems are based on the topics course in admissible sets.

C5. Let α be the first admissible ordinal $> \omega_1$. Show that L_α has property Beta. Conclude that there is a Δ^1_2 ordinal α which is admissible but not recursively inaccessible such that L_α has property Beta. (You may assume any theorem proved in Barwise's book.)

C6. Let $\mathfrak{m} = \langle M, <, p, R, \cdots, R_\ell \rangle$ be a structure where $<$ well-orders M and p is a pairing function. Using the relation between $HYP_{\mathfrak{m}}$ and inductions on $\mathfrak{m}$ prove the following uniformization theorem:

For every inductive relation R there is an inductive relation $S \subseteq R$ such that $dom(R) = dom(S)$ and, for all $x \in dom(R)$

$$\exists ! y \, S(x, y) \quad .$$

D. Set Theory

D1. Prove that $ZF \vdash Con(ZFL-P)$, where ZFL-P is ZF with the axiom of constructibility but not the power set axiom.

D2. Show that forcing with $\mathbb{P}$ collapses $\aleph_\omega$ to ω, where $\mathbb{P}$ is the set of all partial functions $p : \aleph_\omega \to 2$ with $|\text{domain } p| < \aleph_\omega$.

D3. For α a limit ordinal less than ω_1, let C_α be a cofinal ω-sequence in α. Show that there is an uncountable set $X \subset \omega_1$ of limit ordinals such that

$$(\forall \alpha, \beta \in X)(\alpha < \beta \to \alpha \notin C_\beta) .$$

D4. Assume $\Diamond$. Show that there is an $S \subset \omega_1$ such that $\Diamond(S)$ and $\Diamond(\omega_1 \setminus S)$. Hint: Consider the ideal,

$$I = \{S \subset \omega_1 : \text{not } \Diamond(S)\} .$$

D5. Let M be a countable, transitive model of ZFC, let $M[G]$ be a $\mathbb{P}$-generic extension of M where $\mathbb{P}$ is c.c.c. in M. Let $X, Y \in M$. Show that for every $F \in M[G]$ with $F : X \to Y$ there is a $f \in M$ such that for every $x \in X$, we have $f(x) \subset Y$, $(|f(x)| \leq \omega)^M$ and $F(x) \in f(x)$.