Qualifying Exam

LOGIC

August 29, 1985

INSTRUCTIONS: Do four questions, at most two elementary.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

<u>Elementary</u>

1. Prove that there is an $X \subset \mathbb{R} \times \mathbb{R}$ such that for all lines L, $|X \cap L| = 2$. $\mathbb{R}$ is the set of real numbers. <u>Line</u> means straight line - a set of the form $\{(x,y) : ax + by + c = 0\}$, where $a \neq 0$ or $b \neq 0$.

2. Prove that there is a function $f : \omega \to \omega$ such that for all $g \geq f$, K is recursive in g. $g \geq f$ means $g(n) \geq f(n)$ for all n.

3. Find a counter-example to:

   If $A_i \underset{\rightarrow}{\lesssim} B$ and $A_i \prec A_{i+1}$ for all $i \in \omega$, then $\cup_i A_i \underset{\rightarrow}{\lesssim} B$.

Set Theory

1. Let $\kappa = \omega_2$. Assume $L_\alpha \prec L_\kappa$, where $\alpha \leq \omega_1$. Prove that $(\kappa \text{ is inaccessible})^L$. ω_1, ω_2 denote the <u>real</u> ω_1, ω_2, <u>not</u> ω_1^L, ω_2^L.

2. Suppose that for each limit $\gamma < \omega_1$, $A_\gamma \subset \gamma$ and A_γ is a cofinal ω-sequence in γ. Prove that for some uncountable $S \subset \omega_1$, $\{A_\gamma : \gamma \in S\}$ is a Δ-system.

3. Let M be a countable transitive model of ZFC. Let

$$\mathbb{P} = ([\omega]^\omega)^M = \{x \in M : x \subseteq \omega \text{ and } x \text{ is infinite}\} .$$

On $\mathbb{P}$, let $p \leq q$ iff $p \subseteq q$. Let G be $\mathbb{P}$-generic over M. Prove $\mathcal{P}(\omega) \cap M = \mathcal{P}(\omega) \cap M[G]$.

Model Theory

1. Find a counterexample to: If

   a) $\mathcal{A} \cong \mathcal{B}$;

   b) $\langle \mathcal{A}, \bar{a} \rangle \precsim \langle \mathcal{B}, \bar{b} \rangle$; and

   c) $\langle \mathcal{B}, \bar{b} \rangle \precsim \langle \mathcal{A}, \bar{a} \rangle$.

   Then $\langle \mathcal{A}, \bar{a} \rangle \cong \langle \mathcal{B}, \bar{b} \rangle$.

2. Assume that for every countable model $\mathcal{A} \models T$, there is $\bar{a} \in |\mathcal{A}|^{<\omega}$ such that $\langle \mathcal{A}, \bar{a} \rangle$ is homogeneous. Prove that if T has more than countably many countable models up to isomorphism, then it has $2^{\aleph_0}$ countable models up to isomorphism.

3. Prove that if T has countable models, $\mathcal{A}$, $\mathcal{B}$, with $\mathcal{A} \equiv \mathcal{B}$, and neither elementarily embeddable in the other, then T has at least 5 non-isomorphic countable models.

4. Let T be a complete theory and R a binary relation of L(T) such that $T \vdash$ "R is a linear order with no greatest element." Prove that T has an elementary chain of models $M_0 \prec M_1 \prec \cdots$ whose union has a set of indiscernibles $\{a_0, a_1, \cdots\}$ such that $a_n \in M_{n+1}$ and $x \mathrel{R} a_n$ for each $n \in \omega$ and $x \in M_n$.

Recursion Theory

1. Produce a one-one reduction of K to $\{\langle e,x\rangle \mid \langle e,x\rangle \in W_e ; \quad e,x < \omega\}$.

2. Construct a recursive Tree $Tr \subset \omega^{<\omega}$ such that if A is a maximal infinite chain or maximal infinite antichain of Tr , then $O' \leq_T A$.

3. Suppose that φ is a sentence in the language of number-theory augmented by a binary relation symbol. Suppose that for each recursive ordinal λ , there is an $e \in N$ such that $W_e^{(2)}$ well-orders N in type λ and $\langle N,+,\cdot,W_e^{(2)}\rangle \models \varphi$. Prove that for some $e \in N$, $\langle N,+,\cdot,W_e^{(2)}\rangle \models \varphi$ and $W_e^{(2)}$ is not a well-ordering.