answers.

QUALIFYING EXAM Answers
LOGIC

January 20, 1989

ELEMENTARY

1. Prove that for every infinite regular cardinal, κ, there is a cardinal λ such that $\lambda = \omega_\lambda$ and λ has cofinality κ.

Solution. For each ordinal, $\alpha \leq \kappa$, define θ_α by recursion as follows: $\theta_0 = \omega$; $\theta_{\alpha+1} = \omega_{\theta_\alpha+1}$; $\theta_\gamma = sup\{\theta_\alpha : \alpha < \gamma\}$ for limit γ. Let $\lambda = \theta_\kappa$. λ has cofinality κ because it is the supremum of a strictly increasing κ-sequence. Also,

$$\lambda \leq \omega_\lambda = sup\{\omega_{\theta_\alpha} : \alpha < \kappa\} \leq sup\{\theta_{\alpha+1} : \alpha < \kappa\} = \lambda \quad ,$$

so $\lambda = \omega_\lambda$.

2. Suppose T is a consistent theory which has fewer than 2^ω non-isomorphic countable models. Prove that there is a sentence, ϕ, in the language of T, such that $T \cup \{\phi\}$ is complete.

Solution. Suppose there is no such ϕ; then clearly $T \cup F$ is not complete for any finite set, F, of sentences. For $s \in 2^{<\omega}$, define T_s so that

1. $T_\emptyset = T$ ($\emptyset$ is the empty sequence).
2. T_s is consistent and $T \subseteq T_s$.
3. For some sentence, ϕ_s, $T_{s0} = T_s \cup \{\phi_s\}$ and $T_{s1} = T_s \cup \{\neg\phi_s\}$.

(3) is possible because no finite extension of T is complete. Now, for $f \in 2^\omega$, let $T_f = \bigcup_{n\in\omega} T_{f|n}$, and let $\mathcal{A}_f$ be a countable model of T_f. Then these $\mathcal{A}_f$ are not elementarily equivalent, hence not isomorphic. Thus, T has **c** non-isomorphic countable models.

3. Suppose that $f : \omega \to \omega$ and $g : \omega \to \omega$ are recursive functions such that $f(m) < g(n)$ whenever $m < n$. Prove that either the range of f or the range of g (or both) is recursive.

Solution. Assume that the range of f is not recursive; so, of course, it is infinite. Then $i \in ran(g)$ iff

$$\exists n \leq (\mu m(f(m) > i)) \ \ (i = g(n)).$$

MODEL THEORY

1. Let T be the theory, in the binary relation symbol, E, whose models are exactly those structures, $\mathcal{A} = \langle A, E_{\mathcal{A}} \rangle$ such that $E_{\mathcal{A}}$ is an equivalence relation on A. Prove that T is ω-stable.

Solution. It suffices to show that for every ~~countable~~ model $\mathcal{A} = \langle A, E \rangle$ of T and every countable subset B of A, the structure $(\mathcal{A}, b : b \in B)$ has countably many elementary types in one variable. To do this, show that two elements x, y ~~are connected by an automorphism of $(\mathcal{A}, b : b \in B)$ and thus~~ have the same elementary type if either $x = y \in B$, xEb and yEb for some $b \in B$ but $x, y \notin B$, or xEb and yEb fail for all $b \in B$ but the E classes of x and y have the same size. or are both infinite.

2. Prove that transitive closure is not first-order definable, even on finite structures. That is, suppose that the language contains one binary relation symbol, R, and let $\phi(x, y)$ be a formula in two free variables, x, y. Prove that there is a structure, $\mathcal{A} = \langle A, R_{\mathcal{A}} \rangle$ such that A is finite and the transitive closure of $R_{\mathcal{A}}$ is not equal to $\{\langle a, b \rangle : \mathcal{A} \models \phi[a, b]\}$.

Solution. Consider any formula $\phi(x, y)$. For each natural number n, consider the stucture with universe $0, ..., 2n - 1$ such that $R(x, y)$ iff $y = x + 2$. Suppose that ϕ defines the transitive closure of R in each of these structures, so that $\phi(x, y)$ iff $x < y$ and $y - x$ is even. Now use the compactness theorem to get an infinite structure $\mathcal{A}$ which has three elements x, y, z such that $\phi(x, y)$, not $\phi(x, z)$, and there is an automorphism of $\mathcal{A}$ which leaves x fixed and sends y to z, a contradiction.

3. Let $\mathcal{A}$ be any model of Peano arithmetic and κ any cardinal such that $\kappa = \kappa^\omega$. Prove that $\mathcal{A}$ has an elementary extension, $\mathcal{B}$, such that for some $b \in B$, $\{c \in B : \mathcal{B} \models c < b\}$ has size exactly κ. *Warning*: A can have more than κ elements.

Solution. Let D be a nonprincipal ultrafilter over ω. Form an elementary chain $\mathcal{A}_\alpha$, $\alpha < \kappa$, by starting with $\mathcal{A}_0 = \mathcal{A}$, taking unions at limit stages, and taking the ultrapower modulo D at successor stages. Let $b \in A_1$ be the equivalence class of the function $f(n) = n^{\mathcal{A}}$, and for $\alpha > 0$ let $S_\alpha = \{c \in A_\alpha : \mathcal{A}_\alpha \models c < b\}$. S_1 has size 2^ω. S_α strictly increases with α, so S_κ has size at least κ. Using $\kappa = \kappa^\omega$, show by transfinite induction that for each $\alpha \leq \kappa$, S_α has size at most κ. Thus S_κ has size κ as required.

RECURSION THEORY

1. Prove that there are uncountable $X, Y \subset \mathcal{P}(\omega)$ such that for all $x \in X$ and $y \in Y$,

$$\forall c \subseteq \omega((c \leq_T x \wedge c \leq_T y) \rightarrow c \equiv_T \emptyset)$$

Solution. It is enough to produce a perfect set $P \subset 2^\omega$ such that all distinct x, y in P have the desired property; then just take X and Y to be uncountable disjoint subsets of P. P will be the set of all paths through a perfect tree, $T \subset 2^{<\omega}$. Construct T by induction, looking at all pairs, a, b, of Gödel numbers of oracle Turing machines, infinitely often. Make sure that for each such a, b, there are arbitrarily large n such that either: (1) for all paths x, y through T which diverge by level n, $\phi_a^x \neq \phi_b^y$, or (2) (if (1) is impossible) for all paths x, y through T which diverge by level n, ϕ_a^x and ϕ_b^y are recursive.

2. Suppose that $g : \omega \rightarrow \omega$ is a total function and $g \leq_T 0'$. Prove that there is a total recursive $f : \omega \rightarrow \omega$ such that for all $n \in \omega$, $W_{f(n)} \equiv_T W_{g(n)}$.

3. Suppose $A \subseteq \omega$ is r.e. and $A <_T 0'$. Prove tht there are r.e. $B, C \subseteq \omega$ such that:
1) B and C are Turing incomparable.
2) $A <_T B$ and $A <_T C$.
3) $A' \equiv_T B' \equiv_T C'$.

2. By the Limit Lemma fix recursive $h : \omega x \omega \rightarrow \omega$ such that $\forall e[\ g(e) = \lim_{s \rightarrow \infty} h(e,s)\ ]$. Fix a recursive f such that for all x and e:

$$\phi_{f(e)}(x) = \begin{cases} 1 & \text{if } \exists s \geq x[\ \phi_{h(e,s)}(x) \downarrow] \\ \uparrow & \text{otherwise} \end{cases}$$

Then for any n,

$$W_{f(n)} = \text{domain}\ (\ \phi_{f(n)}\ ) =^* \text{domain}\ (\ \phi_{g(n)}) = W_{g(n)}.$$

Since the symmetric difference of $W_{f(n)}$ and $W_{g(n)}$ is finite, certainly $W_{f(n)} \equiv_T W_{g(n)}$.

3. Construct D and E such that $B=A \oplus D$ and $C=A \oplus E$ are the desired sets. As usual in infinite injury arguments, define

$$\hat{\Phi}(e, Y, x, s+1) = \begin{cases} \Phi(e,Y_s,x,s+1) & \text{if } \Phi(e,Y_s,x,s) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

During the construction, meet the following requirements :

$S_{e,B}$: $\hat{\Phi}(e, A \oplus D) \neq E$

$S_{e,C}$: $\hat{\Phi}(e, A \oplus E) \neq D$

to guarantee that B and C are Turing incomparable, and attempt to meet the "pseudo-requirements"

$Q_{e,B}$: $\exists^{\infty} s\ \hat{\Phi}(e, A \oplus D, e, s) \downarrow \ \Rightarrow\ \hat{\Phi}(e, A \oplus D, e) \downarrow$

$Q_{e,C}$: $\exists^{\infty} s\ \hat{\Phi}(e, A \oplus E, e, s) \downarrow \ \Rightarrow\ \hat{\Phi}(e, A \oplus E, e) \downarrow$

to guarantee that $B' \equiv_T A' \equiv_T C'$. Define the S-restraint functions

$$\hat{R}_B(e,x,s) = \begin{cases} \mu t \forall y[\ t \leq y \leq s \Rightarrow \hat{\Phi}(e, A \oplus D, x, y) \downarrow] & \text{if } \hat{\Phi}(e, A \oplus D, x, s) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{R}_C(e,x,s) = \begin{cases} \mu t \forall y[\ t \leq y \leq s \Rightarrow \hat{\Phi}(e,\ A \oplus E\ ,\ x\ ,\ y) \downarrow] & \text{if } \hat{\Phi}(e,\ A \oplus E\ ,\ x\ ,\ s) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

and the Q-restraint functions

$$\hat{r}_B(e,s) = \hat{R}_B(e,e,s) \text{ and } \hat{r}_C(e,s) = \hat{R}_C(e,e,s)\ .$$

At stage s, we will also have $x_{B,s}$ and $x_{C,s}$ that are potential witnesses for meeting the S-requirements. The construction now is similar to the finite injury construction of incomparable r.e. degrees, except that when attempting to add elements to say D at stage s, both $\hat{R}_B$ and $\hat{r}_B$ are respected. The proofs of the appropriate lemmas is also similar, except that they proceed on the "true" stages of the construction. Attempting to meeting $Q_{e,B}$ makes the jump of $A \oplus D$ as low as possible, i.e. A' .

SET THEORY

1. Prove, *without* using the Axiom of Choice, that ω_2 is not the union of countably many countable sets.

Solution. Suppose $\omega_2 = \bigcup_{n \in \omega} A_n$. Define $f : \omega_2 \to \omega \times \omega_1$ as follows: If $\alpha \in \omega_2$, $f(\alpha) = \langle n, \xi \rangle$, where n is least such that $\alpha \in A_n$, and ξ is the order type of $\alpha \cap A_n$. Note that f is 1-1. But this is a contradiction, since even without AC, $\omega \times \omega_1$ has cardinality ω_1.

2. Suppose that $\mathcal{X}$ is a family of ω_1 countable sets and n is a fixed natural number such that $|x \cap y| \leq n$ whenever x and y are distinct members of $\mathcal{X}$. Prove that $\mathcal{X}$ can be written as

$$\mathcal{X} = \{x_\alpha : \alpha < \omega_1\} \quad ,$$

where for each $\alpha < \omega_1$,

$$x_\alpha \cap \bigcup_{\beta < \alpha} x_\beta$$

is finite.

Solution. Say we have,

$$\mathcal{X} = \{y_\alpha : \alpha < \omega_1\} \quad ,$$

where each $y_\alpha \subset \omega_1$. By the Löwenheim-Skolem argument, there is a club, C, of limit ordinals such that whenever $\gamma \in C$, $\alpha < \gamma \Rightarrow y_\alpha \subseteq \gamma$, and $\alpha \geq \gamma \Rightarrow |y_\alpha \cap \gamma| < n$ (this is possible since each n-element subset of γ is contained in at most one y_α. It follows that

n+1

$$y_\alpha \cap \bigcup_{\beta < \alpha} y_\beta$$

is finite whenever α is of the form $\gamma + k$ for $\gamma \in C$ and k finite. Thus, we can get the x_α by re-indexing: Enumerate C as $\{\gamma_\xi : \xi < \omega_1\}$, and let $\{x_{\omega \cdot \xi + k} : k \in \omega\}$ enumerate $\{y_\alpha : \gamma_\xi \leq \alpha < \gamma_{\xi+1}\}$.

3. Assume that M is a countable transitive model of ZFC, $\kappa \in M$, $\mathbf{P}$ is the partial order of finite partial functions from κ into 2, and G is $\mathbf{P}$-generic over M. Let $f \in \omega^\omega \cap M[G]$. Prove that there is a $g \in \omega^\omega \cap M$ such that $\{n : f(n) < g(n)\}$ is infinite.

Solution. $\mathbf{P} = Fn(\kappa, 2)$. Fix a countable (in M) set $I \subseteq \kappa$ such that $f \in M[G \cap Fn(I, 2)$. Let τ be a $Fn(I,2)$-name for f. In M, let $Fn(I,2) = \{p_k : k \in \omega\}$; for each n, choose $g(n)$ big enough so that for all $k < n$, p_k has an extension which forces $\tau(n) < g(n)$. Then no condition can force $\{n : \tau(n) < g(n)\}$ to be finite.