LOGIC QUALIFYING EXAM, JANUARY 1992

INSTRUCTIONS: Do any four problems, including at most two elementary problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

ELEMENTARY PROBLEMS

E1. Answer true or false for each of the following statements. If true, indicate a reason. If false, give a counter-example. α, β, γ range over ordinals.

(Sample). $\forall\alpha\forall\beta\forall\gamma(\alpha\cdot\beta=\beta\cdot\alpha)$.

Answer: False. $\omega\cdot 2\neq 2\cdot\omega$.

a. $\forall\alpha\forall\beta\forall\gamma(\alpha<\beta\Rightarrow\alpha+\gamma<\beta+\gamma)$.

b. $\forall\alpha\forall\beta\forall\gamma(\alpha<\beta\Rightarrow\gamma+\alpha<\gamma+\beta)$.

c. $\forall\alpha\forall\beta\forall\gamma((\alpha+\beta)\cdot\gamma=\alpha\cdot\gamma+\beta\cdot\gamma)$.

d. $\forall\alpha\forall\beta\forall\gamma(\gamma\cdot(\alpha+\beta)=\gamma\cdot\alpha+\gamma\cdot\beta)$.

e. $\forall\alpha\forall\beta\forall\gamma((\alpha+\beta=\beta+\alpha)\wedge(\beta+\gamma=\gamma+\beta)\Rightarrow(\alpha+\gamma=\gamma+\alpha))$.

E2. Let T be a recursive set of sentences in a finite language L. Assume that for each sentence ϕ of L, either $T\cup\{\phi\}$ is inconsistent or $T\cup\{\phi\}$ has a finite model. Prove that the set $\{\phi : T\models\phi\}$ is recursive.

E3. In the language with one unary function symbol f, prove that the theory $\{\forall x f(f(x))=x\}$ has countably many complete extensions, and describe them.

LOGIC QUALIFYING EXAM, JANUARY 1992, MODEL THEORY

M1. Let $D(\mathfrak{A})$ denote the diagram of a model $\mathfrak{A}$, that is, the set of all atomic and negated atomic sentences true in $\mathfrak{A}_A$. Suppose that T is a complete theory, $\mathfrak{A}$ is a model of T, and $T \cup D(\mathfrak{A})$ is complete. Prove that for every elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$, $T \cup D(\mathfrak{B})$ is complete.

M2. Let $\mathfrak{A}$ be an arbitrary model of Peano arithmetic. Prove that $\mathfrak{A}$ has an ultrapower $\mathfrak{B}$ with an element $b \in B$ such that $\{c \in B : \mathfrak{B} \models c \leq b\}$ has size 2^{ω}.

M3. Let T be a complete theory in a countable language and let κ be a cardinal. Prove that T has a countable model $\mathfrak{A}$ and a model $\mathfrak{B}$ of size κ such that every countable elementary submodel of $\mathfrak{B}$ is elementarily embeddable in $\mathfrak{A}$. Hint: Use indiscernibles.

LOGIC QUALIFYING EXAM, JANUARY 1992, RECURSION THEORY

Notation: ϕ_n is the recursive function with Gödel number n and W_n is the domain of ϕ_n. $\leq_T$ means Turing reducible, and $\equiv_T$ means Turing equivalent. B' denotes the jump of B.

R1. a. Show that there is an index n such that $W_n = \{n\}$.
b. Use this and the Padding Lemma to show that

$$K = \{e \mid \phi_e(e) \text{ converges}\}$$

is not an index set.

R2. Given a nonrecursive r.e. set A, give a construction to show that there is a simple set S such that $S \leq_T A$.

R3. Prove that for all sets $A, B \subseteq \omega$, if $A \leq_T B'$ then there is a binary relation $C \equiv_T B$ such that $\lim_s C(s, \cdot) = B$.

LOGIC QUALIFYING EXAM, JANUARY 1992. SET THEORY

S1. Prove that there is a totally ordered set $(X, <)$ of size $\aleph_1$ such that every ordinal $\alpha < \omega_2$ is isomorphic to a subset of X. *Don't* assume CH. *Hint.* Consider $\omega_1^{<\omega}$ ordered lexically, and use induction.

S2. In the following, forcing always refers to the Cohen partial order – that is, finite partial functions from ω into 2. In the ground model, M, assume that F is a *closed* set of real numbers. Prove that the following are equivalent:

1. $1 \Vdash (\check{F} \text{ is closed})$.
2. F is countable in M.

Hint. Uncountable closed sets contain a copy of the Cantor set.

S3. *Notation:* An antichain is a pairwise incompatible family.
Let

$$T = \{s \mid (\exists \alpha < \omega_1)(s : \alpha \to \omega \quad \textit{and} \quad s \quad \textit{is} \quad 1-1)\},$$

ordered by inclusion.

Prove:

a. T has no ω_1-branches.

b. Every uncountable subset of T contains an uncountable antichain.