LOGIC QUALIFYING EXAM, AUGUST 1992

Instructions: Answer any two Elementary problems *plus* any two problems from the area (Set Theory or Model Theory) for which you are signed up.

ELEMENTARY PROBLEMS

E1. Assume GCH. For an infinite cardinal κ, let $F(\kappa)$ be the set of all non-decreasing functions from κ into ω.

(a) Prove that for all infinite $\kappa, \kappa \leq |F(\kappa)| \leq \kappa^+$.

(b) For which κ is $|F(\kappa)| = \kappa$ and for which κ is $|F(\kappa)| = \kappa^+$?

Note: $f : \kappa \to \omega$ is non-decreasing iff $(\forall \alpha < \beta < \kappa)(f(\alpha) \leq f(\beta))$. For example (when κ is uncountable), $f(\alpha) = 3$ for $\alpha < \omega$ and $f(\alpha) = 6$ for $\omega \leq \alpha < \kappa$.

Hint: f can't jump too often.

E2. Let ϕ be a sentence in the language $\{0, +\}$ groups which holds in all divisible torsion free abelian groups. Prove that for all but finitely many primes p, ϕ holds in the cyclic group Z_p of order p.

Note: An abelian group G is divisible if for each integer $n > 0$,

$$G \models (\forall x)(\exists y) ny = x.$$

G is torsion free if for each $n > 0$,

$$G \models (\forall x)[nx = 0 \Rightarrow x = 0].$$

E3. Suppose that there is a model $\mathcal{M} = (M, E)$ for $ZF + \neg Con(ZF)$. Prove that $\mathcal{M}$ is not an ω-model. That is, show that the order type of $\{n \in M : \mathcal{M} \models n \text{ is a natural number}\}$ under the relation E is not ω.

SET THEORY

S1. For each sentence ψ in the language $\{\in, =\}$, let

$$T(\psi) = \{\alpha : 0 < \alpha < \omega_1 \ \&\ L(\alpha) \models \psi\}.$$

Answer "true" or "false" for each of the following implications. If "true", indicate a reason. If "false", describe a ψ which is a counterexample. *Note*: ω_1 means the *real* ω_1 – not necessarily $\omega_1^{(L)}$.

1. $T(\psi) \neq \emptyset$ implies $T(\psi)$ is unbounded in ω_1.
2. $T(\psi)$ is unbounded in ω_1 implies $T(\psi)$ is stationary in ω_1.
3. $T(\psi)$ is stationary in ω_1 implies $T(\psi)$ contains a closed unbounded subset of ω_1.
4. $T(\psi)$ contains a closed unbounded subset of ω_1 implies $T(\psi)$ *is* a closed unbounded subset of ω_1.

S2. Let $\mathcal{P}$ be the partially ordered set consisting of the finite partial functions from ω into $\mathcal{Q}$ (the set of rationals). Let M be a countable transitive model of ZFC, and suppose, *in* M, $\langle \epsilon_n : n \in \omega \rangle$ is a sequence of positive real numbers. Let G be $\mathcal{P}$-generic over M, and in $M[G]$, let $f = \bigcup G : \omega \to \mathcal{Q}$. Show $\bigcup_{n\in\omega}(f(n) - \epsilon_n, f(n) + \epsilon_n)$ contains every real number in M.

S3. Let M be a countable transitive model of ZFC. Suppose, that $\mathcal{P} \in M$ is a partial order such that $\mathcal{P}$ is countable *in M*. Let G be $\mathcal{P}$-generic over M. In $M[G]$, let S be an unbounded subset of $\omega_1^{(M)} = \omega_1^{(M[G])}$. Prove that there is a $T \subseteq S$ such that T is in M and T is unbounded in $\omega_1^{(M)}$.

MODEL THEORY

M1. Prove that the theory of torsion free abelian groups is complete. (See Note following problem **E2**).

M2. If $\mathcal{U}$ is an ultrafilter, and $\mathcal{A}$ is a structure, let $\Pi_{\mathcal{U}}\mathcal{A}$ be the ultrapower of $\mathcal{A}$ modulo $\mathcal{U}$. Let $\mathcal{N}$ be the standard model of arithmetic. Prove that there exist ultrafilters $\mathcal{U}$ and $\mathcal{V}$ (possibly on different index sets) such that the models $\Pi_{\mathcal{U}}(\Pi_{\mathcal{V}}\mathcal{N})$ and $\Pi_{\mathcal{V}}(\Pi_{\mathcal{U}}\mathcal{N})$ are not isomorphic.

M3. Let $\mathcal{A}$ be an uncountable model of Peano arithmetic and let $a \in A$. Prove that $\mathcal{A}$ has an elementary extension $\mathcal{B}$ with a countable sequence of elements $b_n, n \in \omega$, such that

$$\mathcal{B} \models a + n < b_n$$

for all $n \in \omega$, and there is no element $c \in B$ such that

$$\mathcal{B} \models a + n < c < b_n$$

for all $n \in \omega$.