Qualifying Exam
Logic
August 1997

Instructions: If you signed up for Set Theory, do two E and two S problems. If you signed up for Model Theory, do two E and two M problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Given a (nonabelian) group G we say that a linear order $\leq$ on G left-orders G iff for any $x, y, z \in G$ if $x \leq y$ then $zx \leq zy$. Prove that a group G can be left-ordered iff every finitely generated subgroup of it can be left-ordered.

E2. For any set of reals A and B define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

Prove there exists a set of reals A such that $A + A = \mathbb{R}$ but A fails to contain an uncountable closed set of reals. Note that every uncountable closed set has the same size as the set of all reals.

E3. Give an example (with proof) of a theory T in some language $\mathcal{L}$ such that T is not finitely axiomatizable, but is the reduct of a finitely axiomatizable theory. That is, you would have a finitely axiomatizable T' in some language $\mathcal{L}' \supset \mathcal{L}$ such that the models of T are precisely the reducts to $\mathcal{L}$ of the models of T'.

S1. Let X be any infinite set and let $[X]^\omega$ be the countably infinite subsets of X. Show that there exists a function

$$F : [X]^\omega \to [X]^\omega$$

such that for every $A \in [X]^\omega$ there exists $a \in A$ such that $a \in F(A \setminus \{a\})$.

S2. Let $\mathbb{P} = 2^{<\omega}$. Let G be $\mathbb{P}$-generic over M a transitive model of set theory (not necessarily countable). Let $\mathfrak{c}^M$ be the cardinality of the continuum in M. Show that there exists $\langle G_\alpha : \alpha < \mathfrak{c}^M \rangle \in M[G]$ such that

- G_α is $\mathbb{P}$-generic over M for every α and
- $\alpha < \beta$ implies $G_\alpha \neq G_\beta$.

S3. Assume that ZFC has a transitive model. Let ZFC_n be the first n axioms of ZFC.

(a) Show there is a model of "ZFC + Con(ZFC) + ZFC has no transitive model"

(b) Show there is a model of "ZFC + Con(ZFC) + exists n such that ZFC_n has no transitive model"

M1. Let U be a distinguished unary predicate in the language L. An L-structure has type (κ, λ) iff the universe has cardinality κ and the interpretation of U in the structure has cardinality λ. Let $\kappa_0 = \omega$ and for every $n < \omega$ let $\kappa_{n+1} = 2^{\kappa_n}$. Let $\kappa = \sup_{n<\omega} \kappa_n$. Let $\mathfrak{c}$ be the cardinality of the continuum. Assume that at least one of $|L|$ and κ^ω is no more than κ^+. Prove that every L-structure of type $(\kappa, \mathfrak{c})$ has an elementary extension of type $(\kappa^+, \mathfrak{c})$.

M2. Let T be defined as follows:

(a) T has unary predicates P and Q and a three place predicate E, written as yE_xz,

(b) the universe of any model of T is the disjoint union of P and Q, each infinite,

(c) if yE_xz, then $P(x)$, $Q(y)$ and $Q(z)$,

(d) for any fixed x in P, E_x is an equivalence relation on Q with infinitely many equivalence classes, and

(e) if $n < \omega$ and $x_1, \ldots, x_n \in P$ with no repetition, and $y_1, \ldots y_n \in Q$, then for some $y \in Q$ we have that for all $1 \leq l \leq n$ the relation $yE_{x_l}y_l$ holds.

(f) If $n, m < \omega$ and $x_1, \ldots x_n \in P$, while $A_1, \ldots A_m$ are disjoint finite subsets of Q, there is $x \in P$ distinct from $x_1, \ldots x_n$ such that $A_1, \ldots A_m$ are subsets of different E_x equivalence classes.

Note: we obtain a logically equivalent theory if we demand that y in (e) is different than each $y_1, \ldots y_n$.

Show that T has elimination of quantifiers.

M3. Prove that a countable complete theory which has uncountably many types has continuum many pairwise nonisomorphic countable ω-homogeneous models.

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E1. Use the compactness theorem for propositional logic. For each $x, y \in G$, let $p_{x,y}$ be a proposition letter which "says" that $x \leq y$. Then, construct a propositional theory Σ which describes the order. For example, for each $x, y, z \in G$, the sentence $p_{x,y} \Rightarrow p_{zx,zy}$ will be in Σ. You also need some sentences describing the properties of a linear order, such as $p_{x,y} \vee p_{y,x}$ for each x, y. The hypothesis on finitely generated subgroups guarantees that every finite subset of Σ is consistent, and then, a model (i.e., truth assignment) for all of Σ tells you how to order the group.

E2. List the reals as $\{r_\alpha : \alpha < \mathfrak{c}\}$, and list the closed uncountable sets as $\{C_\alpha : \alpha < \mathfrak{c}\}$. Construct A as an increasing union of sets: $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$, where each $|A_\alpha| < \mathfrak{c}$. At limits, take unions, and each $|A_{\alpha+1} \backslash A_\alpha| \leq 2$. Also choose reals, p_α, for $\alpha < \mathfrak{c}$. Given A_α, choose p_α in $C_\alpha \setminus (A_\alpha - A_\alpha)$; then construct $A_{\alpha+1}$ so that $A_{\alpha+1} - A_{\alpha+1}$ contains r_α and $A_{\alpha+1}$ does not contain p_ξ for any $\xi \leq \alpha$.

E3. Let $\mathcal{L} = \emptyset$, so that T is a pure equality theory. Let T say that all its models are infinite. Let $\mathcal{L}' = \{<\}$, and let T' say that $<$ is a total order with no largest element.

S1. Let $A \equiv B$ iff $A \Delta B$ is finite. Choose a representative for each class and let F map each element of the equivalence class to its representative.

S2. Working in M: Let $\{Q_\alpha : \alpha < \mathfrak{c}^M\}$ be a family of pairwise almost disjoint infinite subsets of ω. For each α let $f_\alpha : \omega \to Q_\alpha$ be the increasing enumeration of Q_α.

Then, in $M[G]$: Define $G_\alpha : \omega \to 2$ by $G_\alpha(n) = G(f_\alpha(n))$. Each G_α corresponds, in a natural way, to a filter on $\mathbb{P}$.

S3.

(a) Take any transitive model of least rank.

(b) Note that you can't get a transitive model here (by reflection). However, for each m, let T_m be the theory:

ZFC_m+CON(ZFC)+'$\exists n$ such that ZFC_n has no transitive model'.

Each T_m has a (transitive) model, since (for m large enough), T_m is true in a minimal rank model of ZFC$_m$. The result now follows by the Compactness Theorem.

M1. Take an ω-ultrapower of the model. In the case the language is small take an elementary substructure.

M2. Solution 1: Do the elimination of quantifiers directly, defining an arrangement of $x_1, \dots x_n$ as a formula which specifies which x_i's are P and which Q, which x_i, x_j are equal to each other, and for each x_k in P and x_i, x_j in Q, tells if $x_i E_{x_k} x_j$ or not. Use axioms (d) and (e) in the crucial step of the argument.

Solution 2: Use the notion of model completeness. First observe that T is a consistent theory. Notice that $T_\forall$ is the theory of equivalence relations, which has the amalgamation property. It suffices to show that T is the model completion of $T_\forall$. This can be done by considering all models of $T_\forall$ which are existentially closed for $T_\forall$. Else, we can show that T is model complete by Robinson's test, or we can use a syntactical characterization (see e.g Theorem 3.5.17 in Chang-Keisler's book).

M3. If a countable theory has uncountably many types, then it has continuum many n-types for some n. Every countable model in a countable language extends to a countable ω-homogeneous model, so every n-type is realized in some countable ω-homogeneous model. Since each countable model can realize only countably many types, there must be continuum many non-isomorphic countable ω-homogeneous models.