Qualifying Exam

Logic

August 2001

Instructions:

Do two E and two S problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Find all completions of the theory of one equivalence relation on an infinite set. Prove that all such theories are on your list, none is repeated, and that each is complete and consistent.

E2. Suppose a theory T has infinitely many distinct consistent completions. Show that T has a completion which is not finitely axiomatizable.

E3. Let T be the theory of one infinite, coinfinite unary relation U. Show that T is decidable.

Call $R \subseteq \omega_1 \times \omega$ a *large rectangle* iff $R = A \times B$, where A is an uncountable subset of ω_1 and B is an infinite subset of ω.

S1. Assume $MA + \neg CH$. Fix any $S \subseteq \omega_1 \times \omega$. Prove that S either contains or is disjoint from a large rectangle.

S2. Assume CH. Prove that there is an $S \subseteq \omega_1 \times \omega$ such that S neither contains nor is disjoint from a large rectangle.

S3. Let X_α, for $\alpha \in ON$, be a sequence of sets such that $\alpha < \beta$ implies $|X_\alpha| < |X_\beta|$ and for a limit ordinal α, $|X_\alpha|$ is the sup of $|X_\beta|$ for $\beta < \alpha$. Show that for any α, there is a cardinal $\kappa > \alpha$ such that $|X_\kappa| = \kappa$.

Answers

E1. The compactness theorem requires that any such theory with arbitrarily large finite equivalences classes must have models with any number of infinite classes. On the other hand if there is a bound on the size of the finite classes, then the number of infinite classes is determined (up to infinity) by the theory.

E2. If the language is infinite, then no consistent complete theory is finitely axiomatizable. Hence we may assume there are only countable many sentences in the language of T, say θ_n for $n < \omega$. Construct an increasing sequence

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

such that each T_n has infinitely many distinct consistent completions and $T_{n+1} = T_n \cup \{\theta_n\}$ or $T_{n+1} = T_n \cup \{\neg\theta_n\}$. The theory $\bigcup_{n<\omega} T_n$ is complete and not finitely axiomatizable.

E3. T is $\aleph_0$-categorical and has no finite models, hence it is complete. It is also recursively axiomatizable. Any recursively axiomatizable complete theory is decidable.

S1. Let $\mathcal{U}$ be a non-principal ultrafilter on ω.

Case 1. $C := \{\alpha : S_\alpha \in \mathcal{U}\}$ is uncountable. Then apply $MA + \neg CH$ to the S_α for $\alpha \in C$ to get a large rectangle $A \times B \subseteq S$ with $A \subseteq C$.

Case 2. $D := \{\alpha : S_\alpha \notin \mathcal{U}\}$ is uncountable. Then get a large retangle disjoint from S.

S2. List $[\omega]^\omega$ as $\{x_\xi : \xi < \omega_1\}$. Get $S \subseteq \omega_1 \times \omega$ such that for each α, both S_α and $\omega \backslash S_\alpha$ meet x_ξ for all $\xi < \alpha$.

S3. By hypothesis, $|X_\alpha| \geq \aleph_\alpha$. Given α construct an increasing sequence of cardinals κ_n with $\kappa_0 > \alpha$ and $\kappa_{n+1} > |X_{\kappa_n}|$. If κ is the sup of the κ_n, then $|X_\kappa| = \kappa$.