Qualifying Exam
Logic
January 2002

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let $\mathcal{L}$ be a language containing a single binary relation symbol E, and let $\mathcal{G}$ be an $\mathcal{L}$-structure. An element $x \in \mathcal{G}$ has *finite out-degree* if there are only finitely many y such that $x\,E\,y$ holds in $\mathcal{G}$. Prove that there is no $\mathcal{L}$-sentence φ such that $\mathcal{G}$ satisfies φ if and only if all elements in $\mathcal{G}$ have finite out-degree.

E2. Show in ZFC that there exists a subset A of $\mathbb{R}^2$ that intersects every circle in $\mathbb{R}^2$ in exactly three points.

Hint. You may take the reals as a given and use without proof that there are exactly continuum many closed sets of reals and any uncountable closed set of reals has cardinality the continuum.

E3. Fix a real $x \in (0, 1)$, and assume that the n^{th} bit (past the '.') in the binary representation of x is a computable function of n. Prove that the n^{th} digit in the decimal representation of x is a computable function of n.

Hint. It may be easier to break your proof into two cases, depending on whether or not x is rational.

E4. Show that a set of natural numbers A is finite iff every subset of A is computably enumerable.

Computability Theory

C1. Let $A \subseteq \omega$ be simple. Prove that there exists sets B and C such that
(1) both B and C are simple,
(2) $A = B \cup C$, and
(3) both $A - B$ and $A - C$ are infinite.

C2. Define

$$\Phi_e(x) = \begin{cases} \mu s\ \varphi_{e,s}(x)\downarrow & \text{if } \varphi_e(x) \text{ converges} \\ \infty & \text{otherwise} \end{cases}$$

Prove that for every computable function $g : \omega \to \omega$ there exists a computable $f : \omega \to 2$ such that for every e:
if $\varphi_e = f$ then $\Phi_e(x) > g(x)$ for all but finitely many x.

C3. A learner is a computable mapping $M : \omega^{<\omega} \to \omega$. We say that M learns a total computable function $f : \omega \to \omega$ iff there is an index e such that $\varphi_e = f$ and

$$M(f(0), f(1), \ldots, f(n)) = e \text{ for almost all } n$$

A family S of functions is learnable iff there is a learner M which learns every $f \in S$.

Prove that:

(a) Every computably enumerable family $\{f_0, f_1, ...\}$ of total computable functions is learnable.

(b) The class of all total computable functions is not learnable.

M1. Let F be a field of characteristic zero, and let L be the first-order language with a constant symbol 0, a one-place function symbol f_λ for each $\lambda \in F$ and a two-place function symbol $+$. Let also V be a nontrivial vector space over F, and consider

$$V = (V, +, 0, f_\lambda)_{\lambda \in F}$$

as an L-structure where $+$ is vector addition, 0 is the zero vector, and each $f_\lambda : V \to V$ is scalar multiplication by λ.

1. Show that the theory of V admits quantifier elimination. (You may use any standard facts from Linear Algebra.)

2. Let $S \subseteq V$. Show that the algebraic closure in the model theoretic sense of S in V is equal to the linear subspace of V generated by S.

The algebraic closure in the model theoretic sense of S in V is defined to be the smallest subset A of V such that $S \subseteq A$ and for every first order formula $\varphi(x)$ with parameters from A if there are only finitely many $v \in V$ such that $\varphi(v)$ holds in V, then all of these v are in A.

M2. Let L be a first-order language and T an L-theory, and assume that T is model-complete and universally axiomatizable. Let p be a complete 1-type (over the empty set) consistent with T, and let $\phi(x)$ be an L-formula without parameters with at most one free variable x. The formula $\phi(x)$ isolates p with respect to T if and only if $\phi(x)$ is in p and

$$T \vdash \phi(x) \to \psi(x)$$

for every formula $\psi(x)$ in p. For any L-structure A and any $a \in A$ we denote by $\langle a \rangle$ the substructure of A generated by a.

Show that $\phi(x)$ isolates p with respect to T if and only if for any $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, there is an L-isomorphism $f : \langle a \rangle \longrightarrow \langle b \rangle$ such that $f(a) = b$.

M3. Let L be the language with one binary relation symbol $<$ and one unary operation symbol f. Let T be the L-theory stating that $<$ is a dense linear ordering without endpoints and f is an order preserving bijection such that $f(x) > x$ for all x.

1. Prove that T admits quantifier elimination.
2. Prove that every model of T is o-minimal.
3. Give, with justification, two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ such that the structures $(\mathbb{R}, <, f)$ and $(\mathbb{R}, <, g)$ are models of T, but the structure

   $$(\mathbb{R}, <, f, g)$$

   is not o-minimal.

A structure is o-minimal iff any subset of it which is definable with parameters is a finite union of sets each of which is a point, or an open interval with end points in the structure, or a ray with end point in the structure.

Set Theory

S1. Prove that the following are equivalent:

1. There is a family $\mathcal{F}$ consisting of $\aleph_2$ stationary subsets of ω_1 such that the intersection of any two distinct elements of $\mathcal{F}$ is nonstationary.
2. There is a family $\mathcal{F}$ consisting of $\aleph_2$ stationary subsets of ω_1 such that the intersection of any two distinct elements of $\mathcal{F}$ is countable.

Hint: The diagonal intersection D of a sequence $\{C_\alpha | \alpha < \omega_1\}$ of closed unbounded sets is defined as

$$D = \{\beta < \omega_1 \mid \beta \in \bigcap_{\alpha<\beta} C_\alpha\}$$

Show that D is a closed unbounded set.

S2. Call $\mathcal{H}$ a *MAD family* iff

a. $\mathcal{H} \subseteq \mathcal{P}(\omega_1)$.
b. Each $A \in \mathcal{H}$ is uncountable.
c. $A \cap B$ is countable whenever A, B are distinct elements of $\mathcal{H}$.
d. $\mathcal{H}$ is maximal with respect to (a,b,c).

Let M be a countable transitive model for ZFC, let $\mathbb{P}$ be ccc partial order of M, and let G be $\mathbb{P}$-generic over M. Assume that $\mathcal{H} \in M$ and that $M \models [\mathcal{H}$ is a MAD family$]$. Prove that $M[G] \models [\mathcal{H}$ is a MAD family$]$.

S3. (Do not assume that $V = L$.) Let κ be an uncountable regular cardinal. ZC denotes ZFC minus the Replacement Axiom. Prove that

$$\{\alpha < \kappa : L_\alpha \models ZC \text{ but } L_\alpha \not\models ZFC\}$$

is unbounded in κ but not stationary.

Answers

E1. Let c_n and d be new constant symbols. Let θ_n be the first order sentence saying $c_i \neq c_n$ for $i < n$ and $E(d, c_n)$. Then by the compactness theorem it is easy to check that the set of sentences $\{\varphi\} \cup \{\theta_n : n < \omega\}$ has a model.

E2. Well-order the circles $\{C_\alpha : \alpha < c\}$. Inductively construct increasing $A_\alpha \subseteq R^2$ so that

(1) A_α and no four points of it lie on a circle,

(2) $A_{\alpha+1}$ contains three points of C_α,

(3) $A_{\alpha+1} - A_\alpha$ is finite, and

(4) at limits take unions.

Since three points determine a circle and any two circles intersect in at most two points, it is possible to do (1) and (2).

E3. If x is rational, then the decimal expansion of x is eventually periodic and hence computable. So we may assume that x is irrational. Let

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

where each b_n is 0 or 1 and d_n is $0, 1, \ldots, 9$. Let

$$q_n = \sum_{k=1}^{n} \frac{b_k}{2^k}$$

and suppose we have already computed

$$r_N = \sum_{k=1}^{N} \frac{d_k}{10^k}$$

then we just search for the least n such that for some $i = 0, \ldots 9$

$$r_N + \frac{i}{10^{N+1}} < q_n < q_n + \frac{1}{2^n} < r_N + \frac{i+1}{10^{N+1}}$$

This i must be d_{N+1}. (Note that the above comparison can be made by the usual grade school algorithms for adding fractions and comparing them.)

E4. Suppose A is infinite. Then A contains uncountably many subsets. Since there are only countably many ce sets, one of these must not be ce. On the other hand if A is finite, then all of its subsets are finite and hence ce.

Alternative solution by student on exam. Let $f : \omega \to A$ be a one-to-one, onto, computable function. Let $B = f(\overline{K})$. Then B is not ce, because f shows that $\overline{K} \leq_1 B$.

C1. Let $f : \omega \to A$ be a 1-1 onto effective enumeration. Note that for any simple P, that $P^* = \{f(p) : p \in P\}$ is simple. Find a simple Q whose union with P is ω as follows. Suppose $R \subseteq P$ is an infinite coinfinite computable subset of P. Let π be a computable bijection of ω which swaps R and $\overline{R}$. Let $Q = \pi(P)$. Then P, Q are simple sets whose union is ω and $B = P^*$ and $C = Q^*$ are as needed.

C2. At stage n let $s = g(n)$.

Def $e < n$ is not canceled iff $\forall x < n \;\; \varphi_{e,s}(x) \downarrow \to \varphi_e(x) = f(x)$.

Find the least $e < n$ such that ϕ_e has not been canceled and $\varphi_{e,s}(n) \downarrow$ and put $f(n) = 1 - \varphi_e(x)$.

C3.

(a) Let h be computable so that $f_e = \varphi_{h(e)}$ for all e. On input

$$f(0)f(1)\ldots f(n)$$

the learner searches for the first e such that $f(m) = f_e(m)$ for $m = 0, 1, \ldots, n$ and then outputs $h(e)$.

(b) Assume that M is a learner which learns all computable functions. Start with the empty string σ_0 and extend σ_n inductively to σ_{n+1} such that one obtains an infinite computable sequence on which M does not converge.

Given σ_n, there is a computable function $f \supseteq \sigma_n$ which does not have an index below n. Since M learns f, there is an extension $\sigma_{n+1} \subseteq f$ such that $M(\sigma_{n+1}) > n$.

As one can search for the extension σ_{n+1} effectively only requiring that $M(\sigma_{n+1}) > n$, the whole process gives a computable sequence $\sigma_0, \sigma_1, \ldots$ of strings, each one properly extending the previous one. Therefore, the union of the σ_n is a computable function f such that M outputs arbitrarily large indices while reading f. Contradiction, M does not learn f.

M1.

(1) Let $\exists x \; \phi(x, y_1, \ldots, y_n)$ be a formula such that ϕ is a conjunction of atomic and negation of atomic formulas. By using elementary linear algebra we may assume each of these conjunctions is of the form

$x = \alpha_1 y_1 + \cdots + \alpha_n y_n$ or $x \neq \alpha_1 y_1 + \cdots + \alpha_n y_n$

If the first case ever occurs, then just substitute $\alpha_1 y_1 + \cdots + \alpha_n y_n$ for x in all the others and hence eliminate x. If all the conjunctions are $\neq$ then the formula is equivalent to True.

(2) Suppose $\theta(x, a_1, \ldots, a_n)$ has only finitely many solutions. Then by part (1) it is clear that θ is logically equivalent to saying that x is one of a finite set of linear combinations of the a_i.

M2. Suppose $\phi(x)$ isolates p. Given a, b define $f : \langle a \rangle \longrightarrow \langle b \rangle$ by $f(\tau(a)) = \tau(b)$ where $\tau(x)$ is any term with one free variable. Then since p is complete we have that $\tau(a) = \tau'(a)$ iff $\tau(b) = \tau'(b)$ and so f is well-defined and similarly it is an isomorphism.

Suppose on the other hand that $\phi(x)$ does not isolate p, then there exists $M \models T$, $N \models T$, $a \in M$ and $b \in N$ such that $M \models \phi[a]$ and $N \models \phi[b]$, where p is the type of a in M but not the type of b in N. Since $\langle a \rangle$ and $\langle b \rangle$ are elementary substructures there can be no isomorphism f taking a to b.

M3.

(1) Let $\exists x\ \phi(x, y_1, ..., y_n)$ be a formula such that ϕ is a conjunction of atomic and negation of atomic formulas. Temporarily add the symbol f^{-1} to the language. By using the properties of a linear order and f (ie. we can replace $x < f(x)$ by True) these conjunctions can be taken to be of the form

$x = f^n(y_i)$, $x < f^n(y_i)$ or $x > f^n(y_i)$ where n is an integer (possibly negative or zero).

If the "=" case occurs, then we may substitute and eliminate x. If one of the other cases doesn't occur then the formula is equivalent to True. If both of the other cases occur then just replace each pair $x < f^n(y_i)$, $x > f^m(y_j)$ by $f^n(y_i) > f^m(y_j)$. To get rid of negative exponents just apply f repeatedly to both sides of the equation or inequality, e.g. replace $f^{-3}(y_1) = f^2(y_2)$ by $y_1 = f^5(y_2)$, etc.

(2) Each atomic formula defines either a point or ray or empty set or the whole model. Hence by qe every definable set is a finite boolean combination of these.

(3) Let $f(x) = x + 2$ and $g(x) = f(x) + \sin(x)$. Then the set of x where $f(x) = g(x)$ is the set of multiples of π.

S1. Suppose $\langle S_\alpha : \alpha < \omega_2 \rangle$ is a family. satisfying (1). By the hint: for any α, there exists a club C_α such that $S_\alpha \cap S_\beta \cap C_\alpha$ is countable for all $\beta < \alpha$. Then, $\langle S_\alpha \cap C_\alpha : \alpha < \omega_2 \rangle$ satisfies (2).

S2. Suppose $\dot{X}$ is a name for a new set which is forced by q to be uncountable and almost disjoint from all the members of MAD family $\mathcal{H}$. Define

$$S = \{\alpha : \exists p \leq q\ p \Vdash \alpha \in \dot{X}\}$$

Then S is in the ground model and is uncountable. Hence there exists $Y \in \mathcal{H}$

which has uncountable intersection with S. Since the forcing is ccc we can find $\beta < \omega_1$ such that $q \Vdash \dot{X} \cap Y \subseteq \beta$. Now, to get a contradiction, consider any $\alpha \in S \cap Y$ above β.

S3. To prove the set is unbounded: Let γ be the ω^{th} cardinal of L larger than κ. Then L_γ is a model of ZC but not ZFC (because the last ω-sequence of L-cardinals is definable). By elementary substructures and Mostowski collapse there are unboundedly many $\delta < \kappa$ such that L_δ can be elementarily embeded into L_γ.

To prove the set is nonstationary: Let C be set of $\alpha < \kappa$ such that L_α is an elementary substructure of L_κ. Since κ is regular, L_κ, and hence L_α for $\alpha \in C$, is a model for the Replacement Axiom, so none of these L_α can be a model of ZC without being a model of ZFC.