Qualifying Exam
Logic
January 2003

**Instructions:**

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

**E1.** Let $\beta$ be an ordinal. Assume that $\beta = X \cup Y$ and that $X, Y$ both have order type $\alpha$. Prove that $\beta \leq \alpha + \alpha$.

**E2.** Let $\mathcal{L}$ have unary relations $U_n$ for $n \in \omega$. Let $\Sigma$ be the set of axioms which asserts that $U_0 \supseteq U_1 \supseteq U_2 \cdots$ and each $U_n \setminus U_{n+1}$, as well as the complement of $U_0$, is infinite. Prove that $\Sigma$ is complete.

**E3.** Let $\mathcal{S}$ be a uniformly computable list of decidable sets. That is, $\mathcal{S} = \{A_n : n \in \omega\}$, and the map $(m, n) \mapsto \chi_{A_n}(m)$ is computable. Let $\mathcal{B}$ be the set of all finite boolean combinations of elements of $\mathcal{S}$. Prove that there is a decidable set which is not in $\mathcal{B}$.

## Computability Theory

**C1.** Recall that a set $A \subseteq \omega$ is **1-generic** if for every $\Sigma_1$ set $S$ of strings there is a string $\sigma \subseteq A$ such that either $\sigma \in S$ or $\forall \tau \supseteq \sigma (\tau \notin S)$. Let $f$ be an injective computable function from $\omega$ to $\omega$. Show that if $G$ is 1-generic then $f^{-1}(G)$ is also 1-generic.

**C2.** Let $A$ be a c.e. set. Suppose there is a computable function $f$ such that the sets $D_{f(0)}, D_{f(1)}, \ldots$ are pairwise disjoint and all have nonempty intersection with $\overline{A}$ (here $D_k$ denotes the finite set with canonical index $k$). Suppose further that there is a constant $k$ such that $|D_{f(n)}| \leq k$ for all $n$. Prove that $A$ is not simple.

**C3.** Show that the intersection of a simple and a creative set is creative.

## Model Theory

**M1.** Let $\mathcal{M} = (M, <, +, 0, \dots)$ be an o-minimal expansion of a divisible, ordered, abelian group. Show from scratch that $\mathcal{M}$ has definable Skolem functions; that is, for every $n \in \mathbb{N}$ and every definable set $A \subseteq M^{n+1}$, there is a definable function $f : \Pi_n(A) \longrightarrow M$ such that $(x, f(x)) \in A$ for all $x \in \Pi_n(A)$, where $\Pi_n : M^{n+1} \longrightarrow M^n$ denotes the projection on the first $n$ coordinates.

**Hint:** if $a, b \in M$ are such that $a < b$, then one can canonically pick an element from the interval $(a, b)$ by choosing $\frac{1}{2}(a + b)$.

**M2.** Let $\overline{\mathbb{C}} := (\mathbb{C}, +, -, 0, 1)$ be the field of complex numbers, and let $\mathbb{A} \subset \mathbb{C}$ be the set of all algebraic numbers. Given a formula without parameters $\phi(x)$, where $x = (x_1, \dots, x_n)$ denotes the tuple of all free variables in $\phi$, we define

$$\dim \phi(\mathbb{C}^n) := \max\{\dim(a) : \ a \in \phi(M^n), \ \mathcal{M} \succeq \overline{\mathbb{C}}\},$$

where $\dim(a)$ is the pregeometry dimension of the tuple $a$ obtained from the algebraic closure operation. We call a point $a \in \phi(\mathbb{C})$ generic if $\dim(a) = \dim \phi(\mathbb{C})$. Prove that if $n > 1$, $p(x) \in \mathbb{Z}[x]$ and $\phi(x)$ is the formula $p(x) = 0$, then $\phi(\mathbb{A}^n)$ contains no generic point, but $\phi(\mathbb{C}^n)$ does.

**M3.** $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$ denotes the group of integers modulo 6. Prove that the theory of the group $(\mathbb{Z}_6)^\omega$ is decidable.

Answers

E1. This problem was incorrectly stated. For the correct version see the January 2008 exam.

E2. Let $\mathcal{L}_k$ have unary relations $U_n$ for $0 \leq n \leq k$. Let $\Sigma_k$ be the set of axioms which asserts that $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_k$ and each $U_n \setminus U_{n+1}$ (for $n < k$), as well as $U_k$ and the complement of $U_0$, is infinite. Then each $\Sigma_k$ is $\aleph_0$ – categorical, and hence complete.

E3. Since one can effectively list all finite boolean expressions, one can list $\mathcal{B}$ as $\{B_n : n \in \omega\}$, where the map $(m,n) \mapsto \chi_{B_n}(m)$ is computable. Then the diagonal set $D = \{n : n \notin B_n\}$ is decidable and not in $\mathcal{B}$.

C1. Suppose $S \subseteq 2^{<\omega}$ is $\Sigma_1$. Let

$$S' = \{\tau \in 2^{<\omega} \ : \ \exists \sigma \in S \ \ \forall n < |\sigma| \ \ f(n) < |\tau| \text{ and } \tau(f(n)) = \sigma(n)\}$$

Then $S'$ is $\Sigma_1$. Since $G$ is 1-generic there is a $\tau \subseteq G$ with either $\tau \in S'$ or no extension of $\tau$ in $S'$. The corresponding $\sigma \in S$ witnesses the same for $f^{-1}(G)$ and $S$.

C2. Let $p \leq k$ be such that $|D_{f(n)}| = p$ for infinitely many $n$. Then there is a computable function $g : \omega \longrightarrow \omega$ such that $|D_{f(g(n))}| = p$ for all $n$. So we may assume that $|D_{f(n)}| = k$ for all $n$.

For $j = 1, \ldots, k$ let $d_j : \omega \longrightarrow \omega$ be the computable function such that $d_j(n)$ is the $j$th element of $D_{f(n)}$. Note that by hypothesis, for every $m$ there is at most one $n$ such that $m = d_j(n)$ for some $j \in \{1, \ldots, k\}$. Let $\theta$ be the partial computable function

$$\theta(m) := \begin{cases} n & \text{if there exists } j \text{ such that } m = d_j(n), \\ \uparrow & \text{otherwise.} \end{cases}$$

Thus, if $\theta(m)\downarrow$, then $m = d_j(\theta(m))$ for a unique $j = j(m)$. Now let $q \in \{1, \ldots, k\}$ be such that for infinitely many $n$ we have $|\bar{A} \cap D_{f(n)}| = q$. Let $e$ be such that $A = W_e$. Define the partial computable function

$$\psi(m) := \begin{cases} \uparrow & \text{if } \theta(m)\uparrow, \text{ or } \theta(m)\downarrow \text{ and } \phi_e(m)\downarrow, \\ 1 & \text{if } \theta(m)\downarrow \text{ and } |\{j \neq j(m) : \ \phi_e(d_j(\theta(m))\downarrow\}| \geq k - q. \end{cases}$$

Then $\text{dom}\psi \subseteq \bar{A}$ and $\text{dom}\psi$ is infinite.

C3. Let $S$ be simple and $C$ creative with computable $f$ having the property that if $W_e \subseteq \overline{C}$, then $f(e) \in \overline{C} \cap \overline{W_e}$. Given any $e$ effectively enumerate a sequence $e_n$ such that

$$W_{e_0} = W_e \cap S \text{ and } W_{e_{n+1}} = W_{e_n} \cup \{f(e_n)\}.$$

Simultaneously enumerate $S$ and wait for some $f(e_n)$ to appear in $S$. If one does, put $g(e)$ equal to the first that shows up. The function $g$ witnesses the creativity of $C \cap S$. Suppose $W_e \subseteq \overline{C \cap S}$. Then $W_e \cap S \subseteq \overline{C}$. Hence the $f(e_n)$ are distinct and since $S$ is simple, infinitely many are in $S$. Since $g(e) = f(e_n) \in S$, we see that $f(e_n) \notin W_e$. Since all the $f(e_n)$'s are not in $C$, they are all not in $C \cap S$.

M1. Let $n \in \mathbb{N}$ and $A \subseteq M^{n+1}$ definable. For $x \in M^n$ put $A_x := \{y \in M : (x, y) \in A\}$, a subset of $M$. By o-minimality, for each $x \in M^n$ the topological boundary $B_x$ of $A_x$ is definable and finite. Put $a(x) := \min B_x$ for each $x \in \Pi_n(A)$, then $a : \Pi_n(A) \longrightarrow M$ is a definable function. Consider the disjoint definable sets

$$\begin{aligned} A_1 &:= \{x \in \Pi_n(A) : \ a(x) \in A_x\}, \\ A_2 &:= \{x \in \Pi_n(A) : \ \forall y \in A_x (y > a(x))\}, \\ A_3 &:= \Pi_n(A) \setminus (A_1 \cup A_2). \end{aligned}$$

Note that $(-\infty, a(x)) \subseteq A_x$ for all $x \in A_3$. We further partition $A_2$:

$$\begin{aligned} A_{21} &:= \{x \in A_2 : \ (a(x), +\infty) \subseteq A_x\}, \\ A_{22} &:= A_2 \setminus A_{21}. \end{aligned}$$

Define $b : A_{22} \longrightarrow M$ by $b(x) := \min B_x \setminus \{a(x)\}$. We can now define a Skolem function $f : \Pi_n(A) \longrightarrow M$ for $A$:

$$f(x) := \begin{cases} a(x) & \text{if } x \in A_1, \\ a(x) + 1 & \text{if } x \in A_{21}, \\ \frac{a(x)+b(x)}{2} & \text{if } x \in A_{22}, \\ a(x) - 1 & \text{if } x \in A_3. \end{cases}$$

M2. Any tuple $a \in \mathbb{A}^n$ has dimension 0, because $\emptyset$ is the largest algebraically independent subset of $\{a_1, \dots, a_n\}$. On the other hand, there is $a \in \mathbb{C}^n$ such that $\dim(a) = n$: this is proved by induction on $n$, noting that for any $a_1, \dots, a_{n-1} \in \mathbb{C}$, the algebraic closure of $\{a_1, \dots, a_{n-1}\}$ in $\mathbb{C}$ is countable, so that there is $a_n \in \mathbb{C}$ that is not algebraic over $\{a_1, \dots, a_{n-1}\}$.

Now let $n > 1$ and $p(x) \in \mathbb{Z}[x]$, and let $\mathcal{M} \succeq \overline{\mathbb{C}}$ with underlying set $M$. Then for any $a \in M^n$ such that $p(a) = 0$, the latter equation implies that $\dim(a) \leq n - 1$. Hence $\dim \phi(\mathbb{C}^n) \leq n - 1$. Now pick $a' = (a_1, \dots, a_{n-1}) \in \mathbb{C}^{n-1}$ such that $\dim(a') = n - 1$. Since $\mathbb{C}$ is algebraically closed, there is $a_n \in \mathbb{C}$ such that $p(a_1, \ldots, a_n) = 0$. Hence $n-1 \leq \dim(a) \leq \dim \phi(\mathbb{C}^n) \leq n-1$. It follows that $a$ is a generic point of $\phi(\mathbb{C}^n)$, and since $n-1>0$, no point of $\phi(\mathbb{A}^n)$ is generic.

M3. Let $T$ be the theory of abelian groups of exponent 6 (that is, $\forall x(x+x+x+x+x+x=0)$) such that there are infinitely many elements of order 2 and infinitely many elements of order 3. Then every model of $T$ is a direct sum of an infinite abelian group of exponent 2 and an infinite abelian group of exponent 3. So, $T$ is $\aleph_0$ – categorical, and hence complete.

Commented out problems

The following two problems were probably earlier versions of the above which were accidentally left in the TeX code.

**M1′.** Let $\mathcal{L}$ be a first-order language. We say that an $\mathcal{L}$-theory $T$ has definable Skolem functions if for any $\mathcal{L}$-formula $\phi(y,x)$, where $y$ is a finite tuple of variables and $x$ is a single variable, there is an $\mathcal{L}$-formula $\psi(y,x)$ such that

$$\begin{aligned}
&T \models \forall y \exists x \psi(y,x),\\
&T \models \forall y \forall x \forall z((\psi(y,x) \land \psi(y,z)) \to x = z),\\
&T \models \forall y(\exists x \phi(y,x) \to \exists x(\psi(y,x) \land \phi(y,x)).
\end{aligned}$$

Show (without using cell decomposition) that if $T$ is an o-minimal theory extending the theory of divisible, ordered, abelian groups, then $T$ has definable Skolem functions.

**M3.′** Prove that the theory of the group $(\mathbb{Z}/2\mathbb{Z})^\omega$ is categorical in every uncountable cardinal.

**Hint:** Show first that the theory of $(\mathbb{Z}/2\mathbb{Z})^\omega$ in the language $\mathcal{L} = (+, 0)$ admits quantifier elimination.

answer:

M3. Let $\phi(x_1, \dots, x_n, y)$ be a quantifier-free $\mathcal{L}$-formula in which exactly the variables $x_1, \dots, x_n$ and $y$ occur. We need to show that $\psi(x) := \exists y \phi(x_1, \dots, x_n, y)$ is equivalent in the theory $T$ of the $\mathcal{L}$-structure $\mathcal{M} := (\mathbb{Z}/2\mathbb{Z})^\omega$ to a quantifier-free formula. Since $\mathcal{M}$ has characteristic 2, $\phi(x_1, \dots, x_n, y)$ is equivalent in $T$ to either $x_1 + \dots + x_n + y = 0$ or $x_1 + \dots + x_n + y \neq 0$. But for any $a_1, \dots, a_n \in M$, we have

$$a_1 + \dots + a_n + (a_1 + \dots + a_n) = 0,$$

while

$$a_1 + \dots + a_n + (1 + a_1 + \dots + a_n) = 1 \neq 0,$$

where 1 is the element $(1/2\mathbb{Z}, 1/2\mathbb{Z}, \dots)$ (in fact, any element different from 0 will do). So in both cases, $\psi(x_1, \dots, x_n)$ is equivalent to the $\mathcal{L}$-formula $0 = 0$. This proves quantifier elimination.

It follows that $\mathcal{M}$ is strongly minimal: let $\phi(x, y)$ be an $\mathcal{L}$-formula with a single variable $x$ and a tuple of variables $y = (y_1, \dots, y_n)$, and let $\mathcal{M}' \models T$ with underlying set $M'$. By quantifier elimination, $\phi(x, y)$ is equivalent in $T$ to either $x + y_1 + \dots + y_n = 0$ or $x + y_1 + \dots + y_n \neq 0$. In the first case, for every $a \in M^n$ the set $\phi(M, a)$ has exactly one element (because $\mathcal{M}'$ is a group), while in the second case the set $M \setminus \phi(M, a)$ has exactly one element.

Since $T$ is strongly minimal and $\mathcal{L}$ is finite, it follows that $T$ is categorical in every uncountable cardinal by the usual dimension theory for strongly minimal structures (which closely mimics the categoricity argument for algebraically closed fields). More precisely, let $\mathcal{M}_1, \mathcal{M}_2 \models T$ such that $|\mathcal{M}_1| = |\mathcal{M}_2| = \kappa > \omega$. Since $T$ is strongly minimal, the (model-theoretic) algebraic closure relation gives rise to a pregeometry. Let $B_1$ and $B_2$ be maximal algebraically independent subsets of $M_1$ and $M_2$, respectively. Since $\mathcal{L}$ is finite and $|\mathcal{M}_1| = |\mathcal{M}_2| > \omega$, we must have $|B_1| = |B_2| = \kappa$, so there is a bijection $f : B_1 \longrightarrow B_2$. Since maximal algebraically independent subsets are indiscernible, the map $f$ is elementary. Since $M_1 = \text{acl}(B_1)$ and $M_2 = \text{acl}(B_2)$, the map $f$ extends to an isomorphism $\tilde{f} : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$.