Qualifying Exam
Logic
January 2005

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let Σ be the theory of directed acyclic graphs. Prove that Σ is not finitely axiomatizable. Here, $\mathcal{L} = \{E\}$, where E is binary, and $\Sigma = \{\varphi_n : 1 \leq n < \omega\}$, where φ_1 is $\neg\exists x\,[xEx]$, φ_2 is $\neg\exists x\exists y\,[xEy \wedge yEx]$, and φ_n is

$$\neg\exists x_1 \cdots \exists x_n\,[x_1Ex_2 \wedge x_2Ex_3 \wedge \cdots \wedge x_{n-1}Ex_n \wedge x_nEx_1]$$

"Not finitely axiomatizable" means that there is no finite Π in the same $\mathcal{L}$ such that Π and Σ have the same models.

E2. Prove that the Continuum Hypothesis is equivalent to the statement that there is a subset $A \subseteq \mathbb{R}$ of size $\aleph_1$ such that both A and $\mathbb{R}\backslash A$ meet every perfect subset of $\mathbb{R}$. A set is perfect iff it is closed and infinite and has no isolated points.

E3. Let TOWOE be the theory of total orders without endpoints; here $\mathcal{L} = \{<\}$, and TOWOE includes, besides the axioms for total order, the sentences $\forall x\exists y[x < y]$ and $\forall x\exists y[y < x]$. Prove that TOWOE does not admit quantifier elimination; that is, there is a formula $\varphi(x_1, \ldots x_n)$ which is not provably equivalent (from TOWOE) to any quantifier-free formula.

Computability Theory

C1. Prove that there are $m, n \in \omega$ such that $m \neq n$ and $W_m \cap W_n = \{m, n\}$.

C2. Prove there exists an infinite computable subtree $T \subseteq \omega^{<\omega}$ such that T does not contain an infinite computable chain or an infinite computable antichain.

T is a subtree of $\omega^{<\omega}$ means that $\sigma \subseteq \tau \in T$ implies $\sigma \in T$ for every $\sigma, \tau \in \omega^{<\omega}$.

$C \subseteq T$ is a chain iff $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$ for every $\sigma, \tau \in C$.

$A \subseteq T$ is an antichain iff $\sigma \subseteq \tau$ for $\sigma, \tau \in T$ just in case $\sigma = \tau$.

C3. Let $Q_S = \{e \mid |W_e| \in S\}$. Prove that for every finite nonempty set S of positive integers there exists an n such that Q_S is n-c.e.-complete. *Hint.* If $|S| = 1$ then Q_S is 2-c.e.-complete.

$W_e \subseteq \omega$ is the e^{th} c.e. set, in some standard enumeration.

A is 2-c.e. iff there exists c.e. sets $B \supseteq C$ such that $A = B \backslash C$.

A is n-c.e. iff there exists c.e. sets A_i for $i < n$ such that
$A_0 \supseteq A_1 \supseteq \cdots$ and for all x:
$x \in A$ iff $x \in A_0$ and the largest i such that $x \in A_i$ is even.

A is Γ-complete iff $\Gamma = \{B \ : \ B \leq_m A\}$.

M1. Let $\mathcal{L} = \{<\}$ and let $\mathfrak{A}$ be countable and well-ordered in type ω^2. Prove that $\mathfrak{A}$ has a countable saturated elementary extension.

M2. In the complex numbers $\mathbb{C}$, define $\sqrt{z}$ by $\sqrt{re^{i\theta}} = \sqrt{r} \cdot e^{i\theta/2}$ when $-\pi < \theta \leq \pi$. Let Σ be the theory of the resulting structure, using $\mathcal{L} = \{+, \cdot, 0, 1, \sqrt{}\}$. Prove that Σ is not $\aleph_1$–categorical.

M3. Let $\mathcal{L} = \{+, \cdot, <\}$. Define $\mathfrak{A}$ so that $A = \omega$ and $+, \cdot, <$ have their standard meaning. Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω. Then $\mathfrak{A} \prec \mathfrak{A}^\omega/\mathcal{U}$ and the universe of $\mathfrak{A}^\omega/\mathcal{U}$ is $A^\omega/\mathcal{U} = \{[f] : f \in \omega^\omega\}$. Let $\mathfrak{B}$ be the submodel of $\mathfrak{A}^\omega/\mathcal{U}$ consisting of all $[f]$ such that f is first-order definable in $\mathfrak{A}$. Prove that $\mathfrak{A} \prec \mathfrak{B} \prec \mathfrak{A}^\omega/\mathcal{U}$.

Set Theory

S1. Assume $MA(\aleph_1)$. Fix $A, B \subset \mathbb{R}$ with $|A| = |B| = \aleph_1$ and $A \cap B = \emptyset$. Let S and T be countable dense subsets of $\mathbb{R}$. Prove that there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f(A) \subseteq S$ and $f(B) \subseteq T$.

S2. Let M be a countable transitive model for *ZFC*. Let $\mathbb{P}$ be Cohen forcing — that is, finite partial functions from ω to 2. Let $(X; <) \in M$ be a dense total order. Let G be $\mathbb{P}$-generic over M. Prove that in $M[G]$, $(X; <)$ cannot be Dedekind-complete.

A total order is *Dedekind-complete* iff every subset has a greatest lower bound and a least upper bound. Applying this with the empty subset, X must have a smallest and a largest element.

S3. Define the sequence $\langle C_\alpha : \alpha \in S \rangle$ to be *club guessing* iff:

1. $S \subseteq \omega_1$ is a stationary set of limit ordinals.
2. $C_\alpha \subseteq \alpha$ is unbounded in α for each $\alpha \in S$.
3. For every club $E \subseteq \omega_1$ there exists $\alpha \in S$ such that $C_\alpha \subseteq E$.

Define $\langle C_\alpha : \alpha \in S \rangle$ to be *almost club guessing* iff 1,2, and

3′. For every club $E \subseteq \omega_1$ there exists $\alpha \in S$ and there exists $\beta < \alpha$ such that $(C_\alpha \setminus \beta) \subseteq E$.

Prove that if $\langle C_\alpha : \alpha \in S \rangle$ is almost club guessing, then there exists $\beta < \omega_1$ such that $\langle C_\alpha \setminus \beta : \alpha \in S \setminus \beta \rangle$ is club guessing.

Answers

E1. Suppose that Π is finite and Π and Σ have the same models. Since Π is finite, there is a finite n such that $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \vdash \bigwedge \Pi$. Now, let $\mathfrak{A}$ consist of one cycle of length $n+1$. Then $\mathfrak{A} \models \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$ so $\mathfrak{A} \models \Pi$, but $\mathfrak{A} \not\models \Sigma$, a contradiction.

E2. For $\rightarrow$, let A be any Bernstein set. For $\leftarrow$, note that there are $2^{\aleph_0}$ disjoint perfect sets. If they all meet A, then $|A| = 2^{\aleph_0}$.

E3. Let $\varphi(x_1, x_2)$ be $\exists y\,[x_1 < y < x_2]$, and suppose that $\psi(x_1, x_2)$ is quantifier-free. Let $A = (-\infty, 0] \cup [1, \infty) \subset \mathbb{R}$. Then $A \models \neg\varphi[0, 1]$ and $\mathbb{R} \models \varphi[0, 1]$, while $A \models \psi[0, 1]$ iff $\mathbb{R} \models \psi[0, 1]$. It follows that TOWOE $\not\vdash \forall x_1, x_2\,[\varphi(x_1, x_2) \leftrightarrow \psi(x_1, x_2)]$.

C1. First, fix m such that φ_m is total, so that $W_m = \omega$. Now, let $f(x, y) = 0$ whenever $y \in \{x, m\}$, and let $f(x, y)$ be undefined otherwise. By the Recursion Theorem, fix n such that $\varphi_n(y) = f(n, y)$ for all y. Then $W_m \cap W_n = W_n = \{m, n\}$, and $m \neq n$ because $\varphi_m \neq \varphi_n$.

C2. Build the computable tree T level by level, cutting off high enough to diagonalize against computable chains and antichains without making the tree finite.

C3. If S has k many gaps then Q_S is $2(k+1)$-c.e.-complete. When $S = \{r\}$, the proof is: Upper bound: by inspection. Lower bound: Given a 2-c.e. set C, enumerate r many numbers into a c.e. set $W_{f(n)}$ when n enters C, and enumerate more numbers into $W_{f(n)}$ when n leaves C. For the general case, we get Q_S to be $2n$-c.e.-m-complete where n is the number of maximal intervals $[x, y]$ contained in S.

M1. By taking elementary extensions ω times, we get a $\mathfrak{B} \succ \mathfrak{A}$ of order type $(\omega + \mathbb{Z} \cdot \mathbb{Q}) \cdot (\omega + \mathbb{Z} \cdot \mathbb{Q})$; here $\mathbb{Z} \cdot \mathbb{Q}$ means $\mathbb{Q}$ blocks of $\mathbb{Z}$. Now, let $\tau(x)$ be a type over a finite subset of B. Taking elementary extensions ω more times, we get τ realized by some c in some $\mathfrak{C} \succ \mathfrak{B}$ of the same order type. But then we can automorph c back into B, so τ is realized in $\mathfrak{B}$. Thus, $\mathfrak{B}$ is saturated.

One might also do this first in the case where $\mathfrak{A}$ has order type ω, where it's a bit simpler. Then, use the fact that in $(\omega; <)$, you can define the order type ω^2 (as an ordering of pairs). This also shows that the result holds whenever the order type of $\mathfrak{A}$ is any infinite ordinal $\alpha < \omega^\omega$.

The result is false for $\alpha \geq \omega^\omega$. To see this, note that given positive integers $n_1 > n_2 > \cdots > n_k$, there is a formula $\theta(x)$ such that $(\alpha; <) \models \theta(\xi)$ iff $\xi = \eta + \omega^{n_1} + \omega^{n_2} + \cdots + \omega^{n_k}$ for some $\eta < \xi$. Thus, the theory of $(\alpha; <)$ has $2^{\aleph_0}$ 1-types, so it has no countable saturated model.

M2. Σ is not even ω–stable, since $\mathbb{C}$ realizes $2^{\aleph_0}$ types over $\emptyset$. To see this, define $S = \{z : \sqrt{z^2} = z\}$. Note that $e^{i\theta} \in S$ when $-\pi/2 < \theta < \pi/2$ and $e^{i\theta} \notin S$ when $\pi/2 < \theta < 3\pi/2$. For any $f \in \{0,1\}^\omega$, let $\theta_f = \pi \sum_n f(n) 4^{-n}$ and let $z_f = e^{i\theta_f}$. Then $(z_f)^{4^k} \in S$ iff $f(k) = 0$.

Actually, Σ is not κ–stable for any κ, since the formula "$y - x \in S$" defines a total order when restricted to $\mathbb{R}$.

M3. It is sufficient to prove that $\mathfrak{B} \prec \mathfrak{A}^\omega/\mathcal{U}$; then $\mathfrak{A} \prec \mathfrak{B}$ will follow using $\mathfrak{A} \prec \mathfrak{A}^\omega/\mathcal{U}$. Applying the Tarski–Vaught test, it is sufficient to prove that whenever $\mathfrak{A}^\omega/\mathcal{U} \models \exists x \varphi([f_1], \ldots, [f_n], x)$ with $f_1, \ldots, f_n$ definable in $\mathfrak{A}$, there is a definable g such that $\mathfrak{A}^\omega/\mathcal{U} \models \varphi([f_1], \ldots, [f_n], [g])$. Now by Łoś's Theorem, $\mathfrak{A} \models \exists x \varphi(f_1(i), \ldots, f_n(i), x)$ for almost every $i \in \omega$, so define $g(i)$ to be the least $k \in \omega$ such that $\mathfrak{A} \models \varphi(f_1(i), \ldots, f_n(i), k)$.

S1. Let $\mathbb{P}$ be the set of all finite partial functions from $\mathbb{R}$ to $\mathbb{R}$ such that

1. $p(A \cap \mathrm{dom}(p)) \subseteq S$
2. $p(B \cap \mathrm{dom}(p)) \subseteq T$
3. $|p(x) - p(y)| < |x - y|$ whenever $\{x, y\} \in [\mathrm{dom}(p)]^2$.

Use a Δ–system argument to prove that $\mathbb{P}$ has the ccc. Then, meet $\aleph_1$ dense sets to get a filter G with $g = \bigcup G$ such that $A \cup B \subseteq \mathrm{dom}(g)$ and $\mathrm{dom}(g)$ is dense in $\mathbb{R}$. Then, using (3), g extends uniquely to a continuous function on $\mathbb{R}$.

S2. In M, let $\varphi : \mathbb{Q} \to X$ be 1-1 and order-preserving. In $M[G]$, we have $g = \bigcup G : \omega \to 2$, and let $r = \sum_n g(n) 2^{-n}$. Then r is irrational, and there is no $x \in X$ such that $\varphi(\mathbb{Q} \cap (-\infty, r)) < x < \varphi(\mathbb{Q} \cap (r, +\infty))$.

S3. For any $\beta < \omega_1$, set $\mathcal{C}_\beta = \langle C_\alpha \setminus \beta : \alpha \in S \setminus \beta \rangle$. Then clearly (1) still holds for $\mathcal{C}_\beta$, and (2) also holds whenever β is a successor ordinal (otherwise, maybe $\beta \in S$, and then $\beta \in S \setminus \beta$ but $C_\beta \setminus \beta = \emptyset$).

If the result fails, then for each $\delta < \omega_1$, $\mathcal{C}_{\delta+1}$ is not club guessing, so we can choose a club E_δ so that there is no α in S with $\alpha > \delta$ and $(C_\alpha \setminus (\delta + 1)) \subseteq E_\delta$. Then the diagonal intersection, D, is also a club:

$$D = \left\{ \gamma < \omega_1 : \gamma \in \bigcap \{E_\delta : \delta < \gamma\} \right\} \quad .$$

Applying (3′), fix $\alpha \in S$ and $\delta < \alpha$ such that $(C_\alpha \setminus \delta) \subseteq D$. Then

$$(C_\alpha \setminus (\delta + 1)) \subseteq (D \setminus (\delta + 1)) \subseteq E_\delta \quad ,$$

a contradiction.