Qualifying Exam
Logic
August 2005

**Instructions:**

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

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A *theory* is a set of sentences closed under logical inference.

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**E1.** Let $T$ be a theory in the propositional language $\mathcal{L}$ using the propositional variables $p_n$ for $n \in \omega$. Let $\mathcal{S}$ be the set of all $\mathcal{L}$-sentences. Define a relation $\leq$ on $\mathcal{S}$ by: $\varphi \leq \psi$ iff $T \vdash \varphi \to \psi$. Note that $\leq$ is transitive and reflexive, so we may define an equivalence relation on $\mathcal{S}$ by: $\varphi \equiv \psi$ iff $\varphi \leq \psi$ and $\psi \leq \varphi$ iff $T \vdash \varphi \leftrightarrow \psi$, and there is a natural partial order on the set of equivalence classes, $\mathcal{S}/\equiv$.

Find a theory $T$ such that $\mathcal{S}/\equiv$ is isomorphic to the collection of all finite and all cofinite subsets of $\omega$ (ordered by $\subseteq$).

**E2.** Let $\mathfrak{M}$ be a structure for a first-order language $\mathcal{L}$, and let $T$ be a universal $\mathcal{L}$-theory (i.e., $T$ has as axioms a set of universal closures of quantifier-free formulas). Prove that $\mathfrak{M}$ is a model of $T$ iff every finitely generated substructure of $\mathfrak{M}$ is a model of $T$.

To define "finitely generated": If $A \subseteq M$, let $\langle A \rangle$ be the substructre generated by $A$; so, you add to $A$ the interpretations of all constants of $\mathcal{L}$ and then close under the interpretations of all functions of $\mathcal{L}$. Then a substructure $\mathfrak{M}'$ of $\mathfrak{M}$ is *finitely generated* iff there is a finite nonempty subset $A$ of $M$ such that $\mathfrak{M}' = \langle A \rangle$. If $\mathcal{L}$ has only predicate symbols, then "finitely generated" is the same as "finite".

**E3.** Let $A$ be a set totally ordered by $<$, and assume that in $A$, there are no increasing or decreasing $\omega_1$–sequences, *and* no subsets isomorphic to the rationals. Prove that $A$ is countable.

Computability Theory

**C1.** Let

$$Q = \{e : W_e = \{0, 1, 2, \ldots, e\}\} \quad .$$

Prove that for every $C \subseteq \omega$: $\quad C \leq_m Q$ iff $C$ is 2-c.e.

**C2.** A set $A \subseteq \omega$ is *bi-immune* if neither $A$ nor its complement contains an infinite computable subset.

a. Show that there is a bi-immune set $A \leq_T \mathbf{0}'$.
b. Show that there is no bi-immune set which is a finite Boolean combination of computably enumerable sets.

**C3.** Let $\mathcal{E}_2$ be the collection of all 2-c.e. sets. Let $\mathcal{E}$ be the collection of all c.e. sets. View both of these as first-order structures whose only relation is $\subseteq$. Prove that the structures $\mathcal{E}$ and $\mathcal{E}_2$ are not elementarily equivalent; that is, find a first-order sentence (just using $\subseteq$) which is true in one and false in the other.

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A set $C \subseteq \omega$ is 2-c.e. iff there are computably enumerable sets $A$ and $B$ such that $C = A \setminus B$.

**M1.** Let $T$ be a complete first order theory, and let $M$ be a monster model for $T$.

Recall that $T$ eliminates $\exists^\infty$ if for every formula $\phi(x, \bar{y})$ there is a formula $\psi(\bar{y})$ such that for every $\bar{a} \in M$: $M \models \psi(\bar{a})$ iff $\phi(M, \bar{a})$ is infinite. We then denote $\psi$ by $(\exists^\infty x)\phi(x, \bar{y})$.

Let $\bar{a} = a_1, \ldots, a_n$ and $\bar{b}$ be two sequences of elements of $M$. We define the algebraic dimension, $\mathrm{algdim}(\bar{a}/\bar{b})$, as follows: If $n = 1$: $\mathrm{algdim}(a_1/\bar{b})$ is equal to 0 if $a_1 \in \mathrm{acl}(\bar{b})$, and to 1 if $a_1 \notin \mathrm{acl}(\bar{b})$. In the general case:

$$\mathrm{algdim}(a_1, \ldots, a_n/\bar{b}) = \sum_{i=1}^{n} \mathrm{algdim}(a_i \;/\; \bar{b}, a_1, \ldots, a_{i-1}).$$

Notice that $\mathrm{algdim}(\bar{a})$ depends on the order of $\bar{a}$.

Finally, let $\phi(\bar{x}, \bar{y})$ be a formula, and $\bar{b} \in M$ of the same length as $\bar{y}$. Then $\mathrm{algdim}\phi(\bar{x}, \bar{b}) = \max\{\mathrm{algdim}(\bar{a}/\bar{b}) : M \models \phi(\bar{a}, \bar{b})\}$.

Show that if $T$ eliminates the $\exists^\infty$ quantifier, then the algebraic dimension is definable in the following sense: for every formula $\phi(\bar{x}, \bar{y})$ and every $n$ there is a formula $\psi(\bar{y})$ such that for all $\bar{b} \in M$: $M \models \psi(\bar{b})$ iff $\mathrm{algdim}\phi(\bar{x}, \bar{b}) = n$. (Hint: prove first for $\mathrm{algdim}\phi(\bar{x}, \bar{b}) \geq n$)

**M2.** Let $T$ be a theory in a countably infinite language $\mathcal{L}$.

a. Assume that for every finite sub-language $\mathcal{L}' \subset \mathcal{L}$, the theory $T{\restriction}\mathcal{L}'$ is $\omega$-categorical. Prove that $T$ eliminates $\exists^\infty$.

b. Show that if $T$ is $\omega$-categorical then for every finite sub-language $\mathcal{L}' \subset \mathcal{L}$, the theory $T{\restriction}\mathcal{L}'$ is $\omega$-categorical.

c. Give an example showing that the converse to (b) is false.

**M3.** Let $\mathcal{L}$ consist of a single binary relation $R$. Let $T_1$ be the theory of triangle-free symmetric graphs, axiomatised as follows:

$$(\forall x)\neg Rxx \qquad (\forall xy)(Rxy \to Ryx) \qquad (\forall xyz)\neg(Rxy \& Ryz \& Rzx)$$

Let $T_2$ be $T_1$ plus the axiom:

$$(\forall x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1})\Big(\big(\bigwedge_{j<k<n} \neg Ry_jy_k \;\wedge\; \bigwedge_{i<m}\bigwedge_{j<n} x_i \neq y_j\big) \longrightarrow$$

$$(\exists z)\big(\bigwedge_{i<m} \neg Rzx_i \;\wedge\; \bigwedge_{j<n} Rzy_j\big)\Big)$$

for each $n, m \in \omega$.

a. Prove that $T_2$ has a model.

b. Prove that $T_2$ has quantifier elimination.

## Set Theory

**S1.** Assume $V = L$. Let $P$ be a perfect set of real numbers (that is, $P$ is closed and non-empty and has no isolated points). Prove that there are $x, y \in P$ with $x \neq y$ such that $x, y$ have the same $L$–rank; that is, $x, y \in L(\alpha + 1) \backslash L(\alpha)$ for some $\alpha$. You may view real numbers concretely as Dedekind cuts in the rationals.

**S2.** Let $M$ be a countable transitive model of ZFC+CH. Let $\mathbb{P}$ be a poset in $M$ which is countable in $M$. Let $G$ be $\mathbb{P}$-generic over $M$. Prove that in $M[G]$, there is a nonprincipal ultrafilter $\mathcal{U}$ on $\omega$ such that: For every $f \in M[G] \cap \omega^\omega$ there exists $g \in M \cap \omega^\omega$ such that $\{n : f(n) = g(n)\} \in \mathcal{U}$.

**S3.** Assume MA + $\neg$CH. Prove that if $A_j$, for $j < \omega$, are arbitrary sets and $\limsup_{j \in H} A_j$ is uncountable for all infinite $H \subseteq \omega$, then there is an infinite $H \subseteq \omega$ for which $\bigcap_{j \in H} A_j$ is uncountable.

Here, $\limsup_{j \in H} A_j$ is the collection of all $x$ such that $x \in A_j$ for infinitely many $j \in H$.

Answers

**E1.** Let $T = \{p_{n+1} \to p_n \mid n \in \omega\}$. Let $F$ be the family of finite and cofinite subsets of $\omega$. Define a map from $F$ into $S/\equiv$ by mapping $\{n\}$ to $[p_{n-1} \wedge \neg p_n]$ (or to $[\neg p_0]$ for $n = 0$). It is easy to see that this induces a 1–1 map from $F$ into $S/\equiv$, so we only need to check that the map is onto. Let $\varphi$ be any propositional formula, without loss of generality in disjunctive normal form. Then each disjunct is a conjunction of $p_i$'s and $\neg p_j$'s; by $T$, we may assume that there is at most one $p_i$ and at most one $\neg p_j$ (namely, for the largest $i$ and the least $j$ occurring, if any). By $T$, we have $i < j$. Again without loss of generality (by adding more disjuncts), we may assume that the disjunct is of the form $p_i$, $\neg p_0$, or $p_i \wedge \neg p_{i+1}$. Now the pre-image of the latter two are singletons, and the pre-image of $p_i$ is the complement of the pre-image of $\neg p_i$.

**E2.** $\Rightarrow$: If $\mathfrak{M} \models T$ and $T$ is universal, then every substructure of $\mathfrak{M}$ is a model of $T$.

$\Leftarrow$: If $\mathfrak{M} \not\models T$, say $\mathfrak{M} \not\models \varphi[a_1, \ldots, a_n]$, where $\varphi$ is quantifier-free and $\forall x_1, \ldots, x_n \varphi(x_1, \ldots, x_n) \in T$. Then $\langle\{a_1, \ldots, a_n\}\rangle \not\models T$.

**E3.** If $a \leq b$, let $[a, b]$ denote the usual interval, and if $b \leq a$, let $[a, b] = [b, a]$. Define $a \sim b$ iff $[a, b]$ is countable. Then $\sim$ is an equivalence relation, and the fact that there are no increasing or decreasing $\omega_1$–sequences implies that each equivalence class is countable. Next, note that the equivalence classes are densely ordered; that is, if $a < c$ and $a \not\sim c$, then there is a $b$ with $a < b < c$ and $a \not\sim b$ and $b \not\sim c$ Thus, if there's more than one class, we could pick out a subset $S$ isomorphic to the rationals, where $S$ contains 0 or 1 elements from each equivalence class. So, if there is no such subset, there is only one class, so $A$ is countable.

**C1.** It easy to see that $Q$ is 2-c.e. and that the 2-c.e. sets are closed under $\leq_m$. So it is enough to see that if $A$ and $B$ are c.e. then there exists a computable function $f : \omega \to \omega$ such that $f^{-1}(Q) = A \cap \overline{B}$. We first claim that there exists a total computable $g(x, y)$ such that for every $x, y \in \omega$

$$W_{g(x,y)} = \begin{cases} \emptyset & \text{if } x \notin A \\ \{0, 1, 2, \dots, y\} & \text{if } x \in A \text{ and } x \notin B \\ \omega & \text{if } x \in A \text{ and } x \in B \end{cases}$$

To see this define a partial computable function

$$\rho(x, y.u) = \begin{cases} \uparrow & \text{if } x \notin A \\ \downarrow = 0 & \text{if } x \in A \text{ and } u \leq y \\ \downarrow = 0 & \text{if } x \in A \text{ and } x \in B \end{cases}$$

and use the S-n-m theorem to get $\psi_{g(x,y)}(u) = \rho(x, y.u)$. By the uniform version of the recursion theorem there exists a total computable $f$ such that $W_{f(x)} = W_{g(x,f(x))}$ for every $x$. But then $x \in A \cap \overline{B}$ iff $f(x) \in Q$.

**C2.** a. Note that any infinite c.e. set contains an infinite computable subset so we may replace computable by c.e. in the definition of bi-immune. Define the characteristic function of the set $A$ by a finite-extension oracle construction as $\chi_A = \bigcup_{s \in \omega} \sigma_s$ where $\sigma_s \in 2^{<\omega}$. Obtain $\sigma_s$ by recursion on $s$ as follows: Set $\sigma_0 = \langle\ \rangle$. For $s = 2e$, check whether there is an element $x \geq |\sigma_s|$ in $W_e$; if so, let $\sigma_{s+1} \supset \sigma_s$ with $\sigma_{s+1}(x) = 1$; else let $\sigma_{s+1} = \sigma_s$. This will ensure that if $W_e$ is infinite then $W_e \cap A \neq \emptyset$. Similarly ensure $W_e \cap \overline{A} \neq \emptyset$ when defining $\sigma_{2e+2}$.

b. Writing the Boolean combination $X$ of c.e. sets in conjunctive normal form, we may assume that $X = X_0 \cup X_1 \cup \cdots \cup X_n$ where each $X_i$ is the intersection of c.e. and co-c.e. sets. Since the c.e. sets, and the co-c.e. sets, are closed under intersection, we may assume that each $X_i$ is d.c.e., i.e., of the form $Y_i \setminus Z_i$ where $Y_i$ and $Z_i$ are c.e. By further manipulation, we may assume that $Y_0 \supseteq Z_0 \supseteq Y_1 \supseteq \cdots \supseteq Z_n$. Without loss of generality, we may assume that all these sets (except possibly $Z_n$) are infinite. But then either $X$ contains the infinite c.e. set $Y_n \setminus Z_n$, or $\overline{X}$ contains the infinite c.e. set $Z_n$.

Alternative solution for b. Suppose $A_1, \dots, A_n$ are c.e. sets. Construct a descending sequence of infinite c.e. sets, $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$ such that $B_i \subseteq A_i$ or $B_i \subseteq \overline{A_i}$ for each $i$. This is possible since given $B_i$ either $B_i \cap A_{i+1}$ is infinite and we can take $B_{i+1} = B_i \cap A_{i+1}$ or it is finite and we can take $B_{i+1} = B_i \setminus A_{i+1}$. But then it is easy to see that that $B_n$ is contained in or disjoint from any boolean combination of the $A_i$.

**C3.** First note that closure under unions, for $\mathcal{E}$ or for $\mathcal{E}_2$, is expressed by:

$$\forall Y, Z\ \exists X\ [Y \subseteq X\ \wedge\ Z \subseteq X\ \wedge\ \forall W\ [Y \subseteq W\ \wedge\ Z \subseteq W\ \rightarrow\ X \subseteq W]]$$

This is true in $\mathcal{E}$, but false in $\mathcal{E}_2$. To see this, let $A$ be a 2-c.e. set such that $\overline{A}$ is not 2-c.e.; then $\overline{A} = B \cup \overline{C}$ where $B$ and $C$ are c.e. sets, so that $B$ and $\overline{C}$ are both 2-c.e. An example of such an $A$ is any universal 2-c.e. set; e.g., $A = \{(n_0, n_1, m) : (n_0, m) \in U\ \wedge\ (n_1, m) \notin U\}$, where $U$ is a universal c.e. set.

**S1.** Let $\rho(z)$ be the $L$–rank of $z$; so $z \in L(\rho(z)+1)\backslash L(\rho(z))$.

Let $A, B$ be disjoint subsets of $P \setminus \mathbb{Q}$ such that $\sup(A) < \inf(B)$ and $A$ and $B$ are order-isomorphic to $\mathbb{Q}$. Let $f$ be some order-isomorphism from $A$ onto $B$. Fix a countable $\delta$ such that $L(\delta)$ contains $A$, $B$, $f$.

Note that $\sup(X) \in \overline{X} \subset P$ whenever $X \subseteq A$ or $X \subseteq B$; also, $\sup(X) = \bigcup(X)$ if we view reals as lower Dedekind cuts in the rationals, so that $\sup(X)$ gets constructed as soon as $X$ gets constructed.

Let $A^*$ be the set of $x \in \overline{A}\backslash A$ such that $x$ is a limit from the left (i.e., $x = \sup X$ for some $X \in [a]^\omega$). Then $|A^*| = 2^{\aleph_0} = \aleph_1$. For $x \in A^*$, let $\widehat{x} = \{a \in A : a < x\}$; then $x = \sup(\widehat{x})$. Note that $\rho(x) = \rho(\widehat{x})$ provided $\rho(x) > \delta$. Likewise define $B^*$ and $\widehat{y}$ for $y \in B^*$.

Now, fix any $\alpha > \delta$ such that $\alpha = \rho(x)$ for some $x \in A^*$. Let $y = \sup(f(A))$. Then $y \in B^*$ and $\widehat{y} = f(A)$, and $\rho(y) = \rho(\widehat{y}) = \rho(A) = \rho(x)$.

**S2.** In $M[G]$, we still have CH, so list $\omega^\omega$ as $\{f_\alpha : \alpha < \omega_1\}$. Now, inductively choose $x_\alpha$ and $H_\alpha$ for $\alpha < \omega_1$ so that:

1. $H_\alpha$ is an infinite subset of $\omega$.
2. $g_\alpha \in \omega^\omega \cap M$.
3. $f_\alpha(j) = g_\alpha(j)$ for all $j \in H_{\alpha+1}$.
4. $\xi < \alpha \quad \to \quad H_\alpha \subseteq^* H_\xi$.

Assuming this can be done, we choose $\mathcal{U} \supseteq \{H_\alpha : \alpha < \omega_1\}$. To do the construction: $H_0$ can be $\omega$, and the $H_\alpha$ for limit $\alpha$ are no problem, since (3) says nothing there. Given $H_\alpha$, we choose $g_\alpha$ and $H_{\alpha+1} \subseteq H_\alpha$ using the following argument in $M$:

Back in $M$, we have $\mathbb{P}$–names $\dot{H}$ and $\dot{f}$ such that $\mathbb{1} \Vdash \dot{H} \in [\omega]^\omega$ and $\mathbb{1} \Vdash \dot{f} \in \omega^\omega$. Then, using the fact thet $\mathbb{P}$ is countable, we can, in $\omega$ steps, construct a $g \in \omega^\omega$ such that $\mathbb{1} \Vdash |\{j \in \dot{H} : g(j) = \dot{f}(j)\}| = \aleph_0$.

**S3.** Just in ZFC: inductively choose $x_\alpha$ and $H_\alpha$ for $\alpha < \omega_1$ so that:

1. $H_\alpha$ is an infinite subset of $\omega$.
2. $x_\alpha \in A_j$ for all $j \in H_{\alpha+1}$.
3. $\xi < \alpha \quad \to \quad H_\alpha \subseteq^* H_\xi \;\&\; x_\alpha \neq x_\xi$.

Given $H_\alpha$ and $x_\xi$ for all $\xi < \alpha$, we choose $x_\alpha$ and $H_{\alpha+1} \subseteq H_\alpha$ using the fact that $\limsup_{j \in H_\alpha} A_j$ is uncountable. $H_0$ can be $\omega$, and the $H_\alpha$ for limit $\alpha$ are no problem, since (2) says nothing there.

Now, using MA + $\neg$CH (or just $\mathfrak{p} > \aleph_1$ or $\mathfrak{t} > \aleph_1$), choose an infinite $H$ such that $\{\alpha : H \subseteq H_\alpha\}$ is uncountable.

**M1.** We show by induction on $n$ and the length of $\bar{x} = x_1, \ldots, x_m$ that $\text{algdim}\phi(\bar{x}; \bar{b}) \geq n$ is definable.
– If $n = 0$ then this is always true.
– If $n > 0$ and $m = 0$, this is always false.
– If $m, n > 0$ then: $\text{algdim}\phi(\bar{x}; \bar{b}) \geq n$ holds if and only if

$$\exists x_1 \text{“algdim}\phi(x_2, \ldots, x_m; x_1, \bar{b}) \geq n\text{”},$$

or:

$$\exists^\infty x_1 \text{“algdim}\phi(x_2, \ldots, x_m; x_1, \bar{b}) \geq n - 1\text{”}.$$

Indeed, one direction is clear, while for the other, we observe that if the second holds then there exists $a_1 \notin \text{acl}(\bar{b})$ such that

$$\text{algdim}\phi(x_2, \ldots, x_m; a_1, \bar{b}) \geq n - 1 \quad ,$$

whereby $\text{algdim}\phi(x_1, \ldots, x_m; \bar{b}) \geq n$.

As “$\text{algdim}\phi(\bar{x}; \bar{b}) \geq n$” is definable, so is “$\text{algdim}\phi(\bar{x}; \bar{b}) = n$”.

**M2.**

a. Assume first that $T$ is $\omega$-categorical, and $\phi(x, \bar{y})$ is a formula, $|\bar{y}| = n$. For any $M \models T$ and $\bar{b} \in M^n$, whether or not $\phi(M, \bar{b})$ is infinite or not depends solely on $\text{tp}(\bar{b})$. Let:

$$X = \{\text{tp}(\bar{b}) \colon M \models T, \bar{b} \in M^n, |\phi(M, \bar{b})| \geq \omega\} \subseteq S_n(T).$$

By Ryll-Nardzewski's theorem, the space of types $S_n(T)$ is a finite discrete topological space, whereby $X$ is clopen. Since $X$ is clopen there exists a formula $\psi(\bar{y})$ such that $X = [\psi] = \{p \in S_n(T) \colon \psi \in p\}$. We conclude that $\psi(\bar{y})$ is the formula $(\exists^\infty x)\phi(x, \bar{y})$.

Now $T$ needs not be $\omega$-categorical, so let $\mathcal{L}'$ be the set of symbols in $\phi$, and let $T' = T{\restriction}_{\mathcal{L}'}$. Then Every model of $T$ is a model of $T'$ and $T'$ is $\omega$-categorical, so we reduce to the previous case.

b. Use Ryll-Nardzewski: $T$ is $\omega$-categorical if and only if $S_n(T)$ is finite for all $n$.

c. $\mathcal{L} = \{P_n \colon n < \omega\}$, each $P_n$ is a unary predicate symbol. $T$ says that every conjunction of the form $\bigwedge_{n<m} Q_n(x)$ where each $Q_n$ is either $P_n$ or $\neg P_n$ has infinitely many solutions.

**M3.**

a. $T_1$ is the theory of a triangle-free graph. To build a model of $T_2$, start with $M_0$ being a single point with no edges. Assume we have $M_n \models T_1$, and build a structure $M_{n+1} \supseteq M_n$: its underlying set is $|M_n| \cup \{c_{\bar{a},\bar{b}} \colon \bar{a}, \bar{b} \in M\}$. If $\bar{a}, \bar{b} \in M_n$ satisfy the antecedent of the additional axiom then $c_{\bar{a},\bar{b}}$ has an edge with each $b_i$ an no other; otherwise, $c_{\bar{a},\bar{b}}$ has no edges. Verify that $M_{n+1}$ is also a triangle-free graph.

$M = \bigcup_n M_n$ is a model of $T_2$.

b. It suffices to show that if $M, N \models T_2$ have a common substructure $A$, $\bar{a} \in A$, and $\phi(z, \bar{w})$ is a conjunction of atomic formulas and their negations, then:

$$M \models \exists z\, \phi(z, \bar{a}) \Longleftrightarrow N \models \exists z\, \phi(z, \bar{a}).$$

Indeed, $\phi$ is therefore of the form $\bigwedge_{i<m} \neg Rzx_i \wedge \bigwedge_{j<n} Rzy_j$, where $\bar{x}$ and $\bar{y}$ are sub-tuples of $\bar{w}$. Let $\bar{b}$ and $\bar{c}$ be the corresponding sub-tuples of $\bar{a}$. Assume now that $M \models \exists z\, \phi(z, \bar{a})$, i.e., that $M \models (\exists z)\big(\bigwedge_{i<m} \neg Rzb_i \wedge \bigwedge_{j<n} Rzc_j\big)$. As $M$ is triangle-free we must have:

$$M \models \bigwedge_{j<k<n} \neg Rc_jc_k \;\wedge\; \bigwedge_{i<m} \bigwedge_{j<n} b_i \neq c_j.$$

Then the same holds in $A$ and therefore in $N$. Since $N \models T_2$: $N \models (\exists z)\big(\bigwedge_{i<m} \neg Rzb_i \wedge \bigwedge_{j<n} Rzc_j\big)$, whereby $N \models \exists z\, \phi(z, \bar{a})$.