Qualifying Exam
Logic
August, 2007

Instructions:

If you signed up for Computability Theory, do two E and two C problems.
If you signed up for Model Theory, do two E and two M problems.
If you signed up for Set Theory, do two E and two S problems.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let Σ be any first order theory. Let $\mathcal{G}$ be the class of groups which are isomorphic to a subgroup of the automorphism group of some model of Σ. Prove that there is a first-order theory Π in the language of group theory such that $\mathcal{G}$ is the class of all models of Π. *Hint.* This just uses the Compactness Theorem. You don't need to know any theorems about the structure of the groups in $\mathcal{G}$.

E2. Let $\mathcal{L} = \{p\}$, where p is a 3–place predicate. Define the structure $\mathfrak{A}$ for $\mathcal{L}$ by: A is the unit circle in the plane, and $p(a, b, c)$ holds iff a, b, c are all different and the triangle abc is counterclockwise. Prove that the theory of $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$, is decidable.

E3. Prove that if X is an uncountable set of reals, then it contains a subset order isomorphic to the rationals.

Computability Theory

C1. Prove that given any computable function h, there are indices x and y such that $W_x = D_y$ and $y > h(x)$.

Here, D_y is the finite set with canonical index y. An appropriate explicit definition is:

$$y = \sum_{n \in D_y} 2^n \quad ;$$

that is, the binary representation of y is the characteristic function of D_y.

C2. Prove that any m-degree contains either only a single 1-degree or infinitely many (in fact, an infinite, strictly ascending chain of) 1-degrees. *Hint:* Given A, consider $A \oplus A$, $(A \oplus A) \oplus (A \oplus A)$, etc.

C3. Prove that any mitotic c.e. set is autoreducible. (A c.e. set A is *mitotic* if A is the disjoint union of two c.e. sets B and C with $A \equiv_T B \equiv_T C$; a c.e. set A is *autoreducible* if there is a Turing functional Φ such that for all x, $\Phi(A \setminus \{x\}; x) = A(x)$, i.e., $A(x)$ can be computed from A without querying $A(x)$.)

Model Theory

M1. Let $\mathfrak{A}$ be a structure for a countable $\mathcal{L}$, and assume that $\mathcal{L}$ contains a unary predicate U. Assume that $|A| = \aleph_\omega$, $2^{\aleph_0} = \aleph_5$, and $|U_{\mathfrak{A}}| = \aleph_0$. Prove that there is an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ such that $|B| = \aleph_{\omega+1}$ and $|U_{\mathfrak{B}}| = \aleph_6$.

M2. Let Σ be the theory of infinite abelian groups of exponent 6 (that is, $\forall x\, [x^6 = 1]$).

a. Prove that every complete extension of Σ is ω–stable.
b. Which complete extensions of Σ are $\aleph_1$–categorical?

M3. Let $\mathcal{L} = \{<, U\}$, where U is a unary predicate. Let Σ be the axioms which say that the universe is infinite and $<$ is a total order; so Σ does not mention U. It is easily seen (but you're not required to prove this) that Σ has $2^{\aleph_0}$ maximally consistent extensions. Prove that exactly 7 of these extensions have quantifier elimination.

S1. Let R be a binary relation on ω, S a binary relation on some infinite ordinal α, and assume that $R, S \in L$ and that (in V)

$$\exists X \subseteq \alpha\,[(\omega, R) \cong (X, S)] \qquad (*)$$

Prove that $(*)$ is true in L.

S2. Let κ be an uncountable cardinal of countable cofinality. Let $\mathcal{A}$ be any family of finite sets with $|\mathcal{A}| = \kappa$.

a. Prove there exists a set $\mathcal{B} \in [\mathcal{A}]^\kappa$ and a countable set R such that $X \cap Y \subseteq R$ for all distinct $X, Y \in \mathcal{B}$.

b. Give an example of such an $\mathcal{A}$ where the R in Part (a) cannot be taken to be finite; that is, there is *no* $\mathcal{B} \in [\mathcal{A}]^\kappa$ and a finite set R such that $X \cap Y \subseteq R$ for all distinct $X, Y \in \mathcal{B}$ (so, the standard Δ-system lemma fails if we replace $\aleph_1$ with κ).

S3. Let M be a countable transitive model of set theory. For $n \leq \omega$ let $\mathbb{P}_n$ be the poset of finite partial functions from $\omega \times n$ to 2, ordered as usual by reverse inclusion. Prove that there exists $G \subseteq \mathbb{P}_\omega$ such that $G \cap \mathbb{P}_n$ is $\mathbb{P}_n$-generic over M for each $n < \omega$ but G is not $\mathbb{P}_\omega$-generic over M.

Answers

E1. Given a group G, the natural way to build an $\mathfrak{A} \models \Sigma$ with $G \subseteq \text{aut}(\mathfrak{A})$ is to add a unary f_α for each $\alpha \in G$ and add the statements that each f_α is an automorphism, plus the axioms $\exists x\,[f_\alpha(x) \neq f_\beta(x)]$ whenever $\alpha \neq \beta$, plus the axioms $\forall x\,[f_\gamma(x) = f_\alpha(f_\beta(x))]$ whenever $\gamma = \alpha \cdot \beta$. If this fails to be consistent, then by the Compactness Theorem, there is some finite *bad subset* $\{\alpha_1, \ldots, \alpha_n\}$ such that just the axioms involving $\alpha_1, \ldots, \alpha_n$ cause the inconsistency.

Now, let Π contain the axioms for groups, plus that statement that the group contains no bad subsets. So, for each bad subset $\{\alpha_1, \ldots, \alpha_n\}$ of each group G, Π contains the statement:

$$\neg \exists x_1 \cdots x_n \left[\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \;\wedge \bigwedge_{\alpha_i = \alpha_j \alpha_k} x_i = x_j x_k \right]$$

This is due to Rabin, Michael O. Universal groups of automorphisms of models. 1965 Theory of Models (Proc. 1963 Internat. Sympos. Berkeley) pp. 274–284 North-Holland, Amsterdam.

E2. Let $\mathfrak{B} = ([0,1], <)$. Then $\text{Th}(\mathfrak{B})$ is decidable, since it is the theory of dense total order with endpoints, which is $\aleph_0$–categorical. Now, note that $\text{Th}(\mathfrak{A})$ can be reduced to $\text{Th}(\mathfrak{B})$ by viewing S^1 as $[0,1]/\{0,1\}$.

E3. Write $\mathbb{Q}$ as the increasing union of finite sets Q_n for $n \in \omega$, where each $|Q_n| = n$. Inductively choose $\varphi_n : Q_n \to X$ so that

a. φ_n is 1-1 and order-preserving.
b. $\varphi_{n+1} \supset \varphi_n$.
c. $(\varphi_n(q), \infty) \cap X$ and $(-\infty, \varphi_n(q)) \cap X$ are both uncountable for all $q \in Q_n$.
d. $(\varphi_n(p), \varphi_n(q)) \cap X$ is uncountable for all $p, q \in Q_n$ such that $p < q$.
Then $\bigcup_n \varphi_n$ embeds $\mathbb{Q}$ into X.

C1. Define a computable function g by setting

$$W_{g(x)} = \{1 + \max \bigcup_{z \leq h(x)} D_z\}$$

and apply the Fixed-Point Theorem.

C2.
Lemma 1. There exists $B \equiv_m A$ such that $B <_1 B \oplus B$.
Lemma 2. If $C <_1 C \oplus C$, then $C \oplus C <_1 (C \oplus C) \oplus (C \oplus C)$.

Given these two results, let $A_0 = B$ from Lemma 1. Then inductively define $A_{n+1} = A_n \oplus A_n$. By Lemma 2, $A_n <_1 A_{n+1}$ and since $C \oplus C \equiv_m C$

for any C we have that $A \equiv_m A_n$ for any n and the 1-degrees of the A_n are distinct.

Proof of Lemma 1: For any B suppose $B \oplus B \leq_1 B$, then $B \times \omega \leq_1 B$. Suppose f is a computable one-one reduction of $B \oplus B$ to B. Note that $n \in B$ implies $f(2n) \in B$ and $f(2n+1) \in B$

and

$n \in \overline{B}$ implies $f(2n) \in \overline{B}$ and $f(2n+1) \in \overline{B}$.

Iterating this (and since f is one-tone) we can find a computable map h such that $|D_{h(n.k)}| = 2^k$ for each n, k and

$n \in B$ implies $D_{h(n,k)} \subseteq B$

and

$n \in \overline{B}$ implies $D_{h(n,k)} \subseteq \overline{B}$.

Using h it is easy to construct a one-one computable map reducing $B \times \omega$ to B.

But the cylindar set $B \times \omega$ has maximal 1-degree, i.e., if $C \leq_m B$, then $C \leq_1 B \times \omega$. Hence, if Lemma 1 is false, then the m-degree of A contains only the "top" 1-degree.

Proof of Lemma 2: This has a simliar proof. By the above argument there is a computable map h such that $|D_{h(n.k)}| = 2^k$ for each n, k

$n \in C \oplus C$ implies $D_{h(n,k)} \subseteq C \oplus C$

and

$n \in \overline{C \oplus C}$ implies $D_{h(n,k)} \subseteq \overline{C \oplus C}$.

Obtain a one-one computable map f reducing $C \oplus C$ to C as follows. Note that for any n the set of m such that either $2m$ of $2m+1$ is in $D_{h(n,n+1)}$ is at least half the size of this set or 2^n. So given input n choose m to be the least so that $m \neq f(k)$ for any $k < n$ and either $2m$ or $2m+1$ is in $D_{h(n,n+1)}$. Put $f(n) = m$.

This result is due to Young, Paul R. Linear orderings under one-one reducibility. J. Symbolic Logic 31 1966 70–85.

C3. We describe an algorithm for computing $A(x)$ using an oracle for $A \setminus \{x\}$. Note the usual splitting argument gives that we can compute $B \setminus \{x\}$ and $C \setminus \{x\}$ from $A \setminus \{x\}$ (i.e., input $y \neq x$, check if $y \in A$, if it is, recursively enumerate B and C to see which it is in.) Suppose that $A = \{e_1\}^B = \{e_2\}^C$.

Input x.

Using $A \setminus \{x\}$ simulate the computations

(1) $\{e_1\}^{B \setminus \{x\}}(x)$,

(2) $\{e_2\}^{C \setminus \{x\}}(x)$,

and simultanealously, recursively enumerate A and

(3) wait for x to show up in A.

There are two possible outcomes. Either both (1) and (2) converge or (3) converges. This is because if $x \notin A$ then both (1) and (2) converge, since x is in neither B or C. Since B and C are disjoint, in fact, at least one of (1) or (2) converges to the correct value of $A(x)$. So if both converge to the same value, this value is the value of $A(x)$. If both (1) and (2) converge but to different values, then we know that $x \in A$.

This is due to Ladner, Richard E. Mitotic recursively enumerable sets. J. Symbolic Logic 38 (1973), 199–211.

M1. Obtain $\mathfrak{B} = \mathfrak{A}_{\omega_6}$ as the union of an elementary chain. $\mathfrak{A}_0 = \mathfrak{A}$. When $0 < \alpha < \omega_6$, $|A_\alpha| = \aleph_{\omega+1}$ and $|U_{\mathfrak{A}_\alpha}| = \aleph_5 = 2^{\aleph_0}$. At limits, take unions. Make sure that each $U_{\mathfrak{A}_{\alpha+1}}$ properly contains $U_{\mathfrak{A}_\alpha}$, so that $|U_{\mathfrak{B}}|$ will be $\aleph_6$, not $\aleph_5$.

Given $\mathfrak{A}_\alpha$, let $\mathfrak{C} = \mathfrak{C}_\alpha = (\mathfrak{A}_\alpha)^\omega/\mathcal{V}$, where $\mathcal{V}$ is a non-principal ultrafilter on ω. Then $|U_{\mathfrak{C}}| = 2^{\aleph_0} = \aleph_5$ but $|C| = (\aleph_{\omega+1})^{\aleph_0} = (\aleph_\omega)^{\aleph_0} \geq \aleph_{\omega+1}$, so we get $\mathfrak{A}_{\alpha+1}$ by taking an elementary submodel.

M2. Let V_p^λ be the abelian group of exponent p with λ generators. Then the models $\mathfrak{A} \models \Sigma$ of size $\aleph_1$ are of the form $V_2^\kappa \oplus V_3^\lambda$, where $\max(\kappa, \lambda) = \aleph_1$. If $S \in [A]^\omega$, then the automorphisms fixing S have only countably many orbits, so $\mathrm{Th}(\mathfrak{A})$ is ω–stable. If κ or λ are finite, then $\mathrm{Th}(\mathfrak{A})$ is $\aleph_1$–categorical. If κ and λ are infinite, then $\mathrm{Th}(\mathfrak{A})$ is not $\aleph_1$–categorical, since $V_2^{\aleph_0} \oplus V_3^{\aleph_1} \equiv V_2^{\aleph_1} \oplus V_3^{\aleph_0} \equiv V_2^{\aleph_1} \oplus V_3^{\aleph_1}$.

M3. Let Π be one of these extensions. Consider $\mathfrak{A} \models \Pi$, and write U for $U_{\mathfrak{A}}$. If Π makes U finite, then there is no way to distinguish the first element of U from any other element of U by a quantifier-free formula, so $|U|$ is 0 or 1. Repeating this argument with $A \backslash U$, we see that Π specifies each of $|U|$ and $|A \backslash U|$ to be either 0 or 1 or ∞.

If U is infinite, then there exist $a, b \in U$ with $a < b$ and $\exists x\, [a < x < b]$. Since all such pairs a, b from U with $a < b$ satisfy the same quantifier-free formulas, U must be densely ordered. A similar argument shows that U has no first or last element.

Likewise, if $A \backslash U$ is infinite, then $A \backslash U$ is densely ordered with no first or last element. Also, if $U = \{a\}$, then a must be either the first or last element. We now have the following 7 cases:

1. $U = \emptyset$ and $<$ is a dense total order without endpoints.
2. $U = A$ and $<$ is a dense total order without endpoints.
3. $U = \{a\}$, a is the first element, and $<$ is a dense total order without last element.
4. $U = A \backslash \{a\}$, a is the first element, and $<$ is a dense total order without last element.
5. $U = \{a\}$, a is the last element, and $<$ is a dense total order without first element.

6. $U = A\backslash\{a\}$, a is the last element, and $<$ is a dense total order without first element.
7. $<$ is a dense total order without endpoints and U and $A\backslash U$ are both dense in A.

In all cases, the theory is $\aleph_0$–categorical, and quantifier-elimination can be proved by showing that every n–type is axiomatized by its quantifier-free sentences. To see this, use automorphisms in countable models.

S1. In L, define $\mathbb{P}$ to be the set of all finite partial isomorphisms. So, elements of $\mathbb{P}$ are finite partial functions p from ω to ω such that p is an isomorphism from $(\mathrm{dom}(p), R)$ to $(\mathrm{ran}(p), S)$ and $\mathrm{dom}(p) \in \omega$. Let $<$ order $\mathbb{P}$ by proper extension, so the largest element $\mathbb{1} = \emptyset$.

In V: $\mathbb{P}$ is not well-founded, since we can use an isomorphism from (ω, R) into (α, S) to define a decreasing ω–sequence in $\mathbb{P}$. But then, by absoluteness of "well-founded", $\mathbb{P}$ is not well-founded in L either, and in L we can use a decreasing ω–sequence in $\mathbb{P}$ to prove that $(*)$ is true.

S2. Fix uncountable regular θ_n with $\theta_n \nearrow \kappa$.

For Part (a): Since θ_n is uncountable and regular, the standard Δ-system lemma holds at θ_n, so we can choose $\mathcal{C}_n \in [\mathcal{A}]^{\theta_n}$ such that $\mathcal{C}_n$ forms a Δ-system with some finite root R_n. Then, for each n, the sets $X\backslash R_n$ for $X \in \mathcal{C}_n$ are pairwise disjoint, so we may choose $\mathcal{B}_n \in [\mathcal{C}_n]^{\theta_n}$ such that $(X\backslash R_n) \cap \bigcup_{j<n} \bigcup \mathcal{C}_j = \emptyset$ for all $X \in \mathcal{B}_n$. Now, let $\mathcal{B} = \bigcup_n \mathcal{B}_n$ and $R = \bigcup_n R_n$.

For Part (b): Let $\mathcal{A}$ be the family of all sets of the form $n \cup \{\xi\}$ (of size $n+1$) such that $n \in \omega$ and $\theta_n < \xi < \theta_{n+1}$.

S3. Let $\{D_k : k \in \omega\}$ list all sets $D \in M$ such that for some $n < \omega$: $D \subseteq \mathbb{P}_n$ and D is dense in $\mathbb{P}_n$. Say D_k is dense in $\mathbb{P}_{n_k}$.

As in the usual proof of the generic set existence lemma, get a sequence $\mathbb{1} = p_0 \geq p_1 \geq p_2 \cdots \in \mathbb{P}_\omega$, and let $G = \{q \in \mathbb{P}_\omega : \exists k\,[q \geq p_k]\}$. Then G is a filter on $\mathbb{P}_\omega$ and each $G \cap \mathbb{P}_n$ is a filter on $\mathbb{P}_n$. Make sure that $p_{k+1} \restriction (\omega \times n_k) \in D_k$ for each k. Then $G \cap \mathbb{P}_n$ will be $\mathbb{P}_n$-generic for each $n < \omega$. Also make sure that each $p_k(0, \ell) = 0$ whenever $(0, \ell) \in \mathrm{dom}(p_k)$; to ensure this at each stage, get $p_k \geq r_k \geq p_{k+1}$, where $r_k(0, \ell) = 0$ for all $\ell < n_k$. Then G is not generic because it does not meet the dense set $\{q : \exists \ell\,[q(0, \ell) = 1]\}$.