

Instructions:

Do two E problems and two problems in the area C, M, or S in which you signed up.

Write your letter code on **all** of your answer sheets.

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper, include your secret letter code and which logic exam you are taking. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let T be a theory in the language consisting of a single binary relation symbol such that T has an infinite model which is an equivalence relation. Prove that T has two isomorphic countable models \mathfrak{A}_0 and \mathfrak{A}_1 such that \mathfrak{A}_0 is a *proper* elementary submodel of \mathfrak{A}_1 .

E2. Let $(I, <)$ be a totally ordered set with $|I| = \aleph_1$. Let \mathcal{W} be the family of all subsets of I that are well-ordered by $<$. Prove that $|\mathcal{W}|$ is either \aleph_1 or 2^{\aleph_0} or 2^{\aleph_1} .

E3. Let T be a recursively axiomatizable complete theory in an infinite language. Prove that T has a recursive set of axioms which is independent. Recall that a set of axioms is independent if no one of them is a logical consequence of the others.

Computability Theory

C1. A set S is superlow if S' is truth-table (tt) below $0'$. Directly construct a nonrecursive superlow set.

Recall that truth-table reducible means Turing reducible using a Turing functional which is total for all possible oracles.

C2. Given a 2-coloring c of pairs of natural numbers (i.e., $c : [\omega]^2 \rightarrow 2$), an infinite set S is a “linear set” for the coloring c if $S = \{a_0 < a_1 < a_2, \dots\}$ and the values $c(\{a_i, a_{i+1}\})$ for all i are the same.

- (1) Is there a computable 2-coloring for which there is no Σ_1 linear set?
- (2) Is there a computable 2-coloring for which there is no Δ_2 linear set?

C3. Let $P \subseteq 2^\omega$ be a nonempty Π_1^0 class. Show that if A is not c.e., then there is an $X \in P$ such that A is not c.e. relative to X .

Set Theory

S1. Assume that (G, \cdot) is a group which is an element of Gödel's constructible universe, L , and in L (G, \cdot) has no subgroup isomorphic to the rationals, $(\mathbb{Q}, +)$. Prove that in the universe, V , it's also true that (G, \cdot) has no subgroup isomorphic to $(\mathbb{Q}, +)$.

S2. Let M be a countable transitive model of ZFC, and let \mathbb{P} be Cohen forcing (finite partial functions from ω to ω). Call a function $f : \omega \rightarrow \omega$ Cohen generic over M iff the filter $G_f = \{p \in \mathbb{P} : p \subset f\}$ is \mathbb{P} -generic over M in the usual sense, it meets every dense subset of \mathbb{P} which is in M . Suppose $f \in \omega^\omega$ is \mathbb{P} -generic over M and $g \in \omega^\omega$ is in M and one-to-one. Define $h(n) = f(g(n))$. Prove that h is \mathbb{P} -generic over M . Prove it's not true if g is only assumed to be finite-to-one.

S3. Suppose $(A_\alpha \subseteq \omega_2 : \alpha < \omega_2)$ and there exists $\alpha_0 < \omega_2$ such that all have A_α order type α_0 .

- (a) Prove there exists $S \subseteq \omega_2$ unbounded in ω_2 and $\delta_0 < \omega_2$ such that $A_\alpha \cap A_\beta \subseteq \delta_0$ for all $\alpha \neq \beta \in S$.
- (b) Give an example showing that there may be no stationary S as in (a).

Sketchy Answers or Hints

E1. First show that T has a infinite countable model \mathfrak{A}_0 which is an equivalence relation and either

- it has infinitely many infinite classes or
- for some n it has exactly n infinite classes and for some m all classes of size greater than m are infinite.

Show that any countable elementarily extension of \mathfrak{A}_0 is isomorphic to it.

Remark: This is true for any theory in a countable language with an infinite model.

E2. Intended solution: Obviously, $|\mathcal{W}| \leq 2^{\aleph_1}$. If I contains an uncountable well-ordered set, then $|\mathcal{W}| = 2^{\aleph_1}$.

If not, then $|\mathcal{W}| \leq (\aleph_1)^{\aleph_0} = 2^{\aleph_0}$, and if \mathcal{W} contains an infinite well-ordered set, then $|\mathcal{W}| \geq 2^{\aleph_0}$, so $|\mathcal{W}| = 2^{\aleph_0}$.

If all well-ordered subsets of I are finite (i.e., $<$ is an inverse well-order), then $|\mathcal{W}| = |I| = \aleph_1$.

E3. Given a list of axioms $\{\psi_n : n < \omega\}$ for T construct a strictly increasing sequence of axioms for T , $\{\theta_n : n < \omega\}$ as follows. Let θ_0 be any consequence of T which is not a logical validity. Given θ_n let $\theta = \theta_n \wedge \psi_n$. Using that the language is infinite and T is decidable, show that one may effectively find ρ a consequence of T such that $\theta \wedge \neg\rho$ is consistent. Let $\theta_{n+1} = \theta \wedge \rho$. Show that the sequence

$$\theta_0, \theta_0 \rightarrow \theta_1, \theta_1 \rightarrow \theta_2, \dots$$

is an independent axiomatization of T .

C1. Construct S by initial segments, forcing the jump. Let σ_0 be the empty string. At stage $s + 1$, if there is a τ extending σ_s such that $\Phi_s^\tau(s) \downarrow$, then let σ_{s+1} be the first such τ found in some standard search. Otherwise, let $\sigma_{s+1} = \sigma_s$.

Note that $S = \bigcup \sigma_s$ is Δ_2^0 and, because we forced the jump, low. In fact, we will argue that it is superlow. Note that if we have $S' \upharpoonright s$, then we can effectively find σ_s . But $S'(s)$ is Σ_1^0 given σ_s . We will use this to recursively determine S' from \emptyset' .

Define a computable function $f: 2^{<\omega} \rightarrow \omega$ such that $\phi_{f(\rho)}(\cdot)$ first attempts to construct $\sigma = \sigma_{|\rho|}$ assuming that ρ is a prefix of S' . It then searches for a τ extending σ such that $\Phi_{|\rho|}^{\tau}(|\rho|) \downarrow$. If the search is successful, then $\phi_{f(\rho)}(\cdot) \downarrow$. Using f we can recursively compute S' from \emptyset' using the fact that

$$S'(s) = \emptyset'(f(S' \upharpoonright s)).$$

This gives us a truth-table reduction.

C2. Say the computable two coloring c colors every pair either red or blue.

1. We construct a computable coloring c with no Σ_1^0 linear set. At stage s , we define c on all pairs $m < s$.

We first describe a module to satisfy the requirement

$$R_e: W_e \text{ is not a linear set for } c.$$

Say that the module is initialized at stage t_0 . At stage s , let $a, b \in W_e$ be the least consecutive pair such that $b > a \geq t_0$ and assume that the pair first appears in W_{e,t_1} , where we can assume that $t_1 \geq b$. The module demands that $c(m, s)$ is different from $c(a, b)$ for every $m \in [b, t_1)$. Note that if a, b have reached their final values, so that this holds for every $s \geq t_1$, then the module ensures that W_e is not a linear set for b . To see this, let m be the largest element in $W_e \cap [b, t_1)$ and s the smallest element in $W_e \cap [t_1, \infty)$. Then $c(a, b)$ has a different color than $c(m, s)$.

Organize the modules on a (finite injury) priority tree, where the nodes at level e are modules for R_e . The outcome of a module is the pair (a, b) . The priority tree ensures that the active modules are not making incompatible demands because if a module thinks that its outcome is (a, b) and these entered W_e at stage t_1 , then all lower priority requirements visited were initialized on or after stage t_1 , so they only make demands on pairs with least element $> t_1$.

2. We show that there is always a Δ_2^0 linear set. Call n a *red point* if $c(n, m)$ is red for every $m > n$. Note that \emptyset' can determine if n is a red point. If there are infinitely many red points, then this set is linear (and in fact, homogeneous). If not, we claim that there is a *computable* linear set for blue. Let a_0 be larger than every red point. If we have defined a_i , then using the fact that it cannot be a red point, let $a_{i+1} > a_i$ be least such that $c(a_i, a_{i+1})$ is blue. Then $\{a_0 < a_1 < a_2 < \dots\}$ is a linear set.

C3. We force by nonempty Π_1^0 -classes (actually subclasses of P). Given a Π_1^0 class P_e and a requirement W_e (the e -th enumeration oracle machine), we first ask:

Is there an x such that $A(x) = 0$ and there is an n such that up to level n we see that every member Y of P_e has $W_e^Y(x) = 1$?

If so, the requirement is automatically satisfied (and $P_{e+1} = P_e$). If not, then we ask:

Is there an x such that $A(x) = 1$ and $\{Y \in P_e : W_e^Y(x) = 0\}$ is nonempty?

If so, take the above class as P_{e+1} (in other words, terminate all the nodes which enumerate x via W_e). If the answers to both questions are negative, then it is easy to see that we can enumerate A from P_e directly, since $A(x) = 1$ if and only if there is an n such that up to level n we see that every member Y of P_e has $W_e^Y(x) = 1$.

S1. Intended solution: Working in L , list \mathbb{Q} as $\{q_i : i < \omega\}$, and then construct a tree T of height ω such that any infinite branch through T would yield a subgroup of G isomorphic to \mathbb{Q} . Then, use the fact that well-foundedness is absolute.

Remark: If the group happened to have been countable in L , then this would be consequence of Shoenfield's Absoluteness.

S2. If $D \subseteq \mathbb{P}$ is dense then show that

$$\{p \in \mathbb{P} : p \circ g \in D\}$$

is dense.

S3. (b): take A_α of order type ω_1 and $A_\alpha \cap \alpha$ unbounded in α for limit α . for $\alpha > \delta_0$ take $\delta_\alpha \in A_\alpha$ with $\delta_0 < \delta_\alpha < \alpha$ and apply push-down.