Instructions:

Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Let M be a structure where $\phi(x, y)$ defines a linear order on an infinite set $X \subseteq M$. Given any linear order-type τ, show that there are an $N \succeq M$ and an infinite set $Y \subset N$ of order-type τ defined by $\phi(x, y)$.

E2. Let $\oplus$ and $\otimes$ both be computable functions from $\omega \times \omega$ into ω such that $(\omega, \oplus)$ and $(\omega, \otimes)$ are both abelian groups with the property that every element other than the identity has order 5 or 7 or 35. Assume also that the two groups are isomorphic. Prove that there is a computable isomorphism between them.

E3. Prove that there is no family $\{A_\alpha : \alpha < \omega_1\} \subset \mathcal{P}(\omega)$ such that for all $\alpha < \beta$: $A_\beta \backslash A_\alpha$ is infinite and $|A_\alpha \backslash A_\beta| \leq 7$.

[1] Note that this is different from exams up until a year ago.

Computability Theory

C1. Consider the set $\mathrm{SD} = \{e \colon \varphi_e(0)\downarrow \text{ and } (\forall j < e)\,[\varphi_j(0) \neq \varphi_e(0)]\}$. Prove that SD is *immune*, i.e., contains no infinite c.e. set.

C2. Let P be a nonempty Π^0_1-class. For any set A, show that there is a $B \in P$ such that the infimum of $\deg_T(A)$ and $\deg_T(B)$ is $\mathbf{0}$.

C3. For any set A, construct a set $B \leq_T A''$ such that $\Sigma^0_1 = \Sigma^0_1[A] \cap \Sigma^0_1[B]$.

Sketchy Answers or Hints

E1 ans. Introduce new constant symbols c_x for $x \in \tau$ and add to the theory all sentences of the form $\phi(c_x, c_y)$ for $x < y$ in τ. Now use Compactness.

E2 ans. Using the group operation $\oplus$: Let $H_5, H_7 \subseteq \omega$ be the subgroups consisting of the identity together with all elements of order $5, 7$ respectively. Likewise get $K_5, K_7 \subseteq \omega$ using the group operation $\otimes$. Then $(H_5, \oplus) \cong (K_5, \otimes)$, and let $f : H_5 \to K_5$ be a computable isomorphism. To get f: View $(H_5, \oplus)$ as a vector space over the 5 element field, and choose (by recursion) a computable basis $A_5 \subseteq H_5$. Likewise let $B_5 \subseteq K_5$ be a computable basis for $(K_5, \otimes)$. Note that $0 \leq |A_5| = |B_5| \leq \aleph_0$. Then a computable bijection from A_5 onto B_5 generates a computable isomorphism from $(H_5, \oplus)$ onto $(K_5, \otimes)$. Likewise $(H_7, \oplus) \cong (K_7, \otimes)$, and let $g : H_7 \to K_7$ be a computable isomorphism. Then f, g generate a computable isomorphism from $(\omega, \oplus)$ onto $(\omega, \otimes)$; To see this, note that $(\omega, \oplus)$ is the direct sum of $(H_5, \oplus)$ and $(H_7, \oplus)$, and $(\omega, \otimes)$ is the direct sum of $(K_5, \otimes)$ and $(K_7, \otimes)$.

E3 ans. Assume that we had such $\{A_\alpha \colon \alpha < \omega_1\}$. For each ξ, choose $B_\xi \subset A_{\xi+1} \backslash A_\xi$ with $|B_\xi| = 8$. Since $|[\omega]^8| = \aleph_0$, fix ξ, η such that $\xi < \xi + 1 < \eta < \eta + 1$ and $B_\xi = B_\eta$. Let $B = B_\xi = B_\eta$. Then $B \subseteq A_{\xi+1}$ and $B \cap A_\eta = \emptyset$, so $B \subseteq A_{\xi+1} \backslash A_\eta$, so $|A_{\xi+1} \backslash A_\eta| \geq 8$, which is a contradiction (taking $\alpha = \xi + 1$ and $\beta = \eta$).

C1 ans. Towards a contradiction, fix an infinite c.e. subset W of SD. By the Recursion Theorem, we can fix an index e for which we can control $\varphi_e(0)$. Given e, wait for an index $i > e$ to enter W and then set $\varphi_e(0) = \varphi_i(0)$, a contradiction.

C2 ans. For each pair e, i of indices, try to force $\exists x\, [\Phi_e^A \neq \Phi_i^X]$ for all X in a nonempty Π^0_1-subclass; otherwise the common value (if total) can be computed effectively.

C3 ans. Let B be a 2-generic relative to A. Suppose for some e and i, some condition σ forces that $W_e^A = W_i^B$. Then consider the set W of all x

so that for some $\tau \supseteq \sigma$, $x \in W_i^\tau$. If there is some $x \in W \setminus W_e^A$, then there is a $\tau \supseteq \sigma$ so that τ forces $x \in W_i^B$, which is a contradiction, since τ also forces that $W_e^A = W_i^B$. If not, then $W_i^B \subseteq W \subseteq W_e^A$. Thus if $W_e^A = W_i^B$, then they are equal to the c.e. set W.