Instructions: Do all six problems.[1]

If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

If you are unable to solve a problem completely, you may receive partial credit by weakening a conclusion or strengthening a hypothesis. In this case, include such information in your solution, so the graders know that you know that your solution is not complete.

If you want to ask a grader a question during the exam, write out your question on an $8\frac{1}{2}$ by 11 sheet of paper. Give it to the proctor. The proctor will contact one of the logic graders who will retrieve your written question, write a response, copy the sheet of paper, and return it to the proctor.

E1. Say that a linear order is an *almost well-order* if every proper final segment of it is well-ordered. For example, ω^* is an almost well-order but not a well-order. Prove that there are continuum many (non-isomorphic) countable almost well-orders.

E2. Let $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ be a sequence of L theories such that for each $n \in \omega$ there exists a model of T_n that is not a model of T_{n+1}. Prove that $\bigcup_{n\in\omega} T_n$ is not finitely axiomatizable. If L is finite, prove that $\bigcup_{n\in\omega} T_n$ has an infinite model.

E3. Prove or refute: There exists a consistent recursively enumerable $T \supseteq PA$ so that $T \vdash \neg\text{con}(T)$ (note that the formula $\text{con}(T)$ depends on the enumeration of T).

i.e. There is a consistent theory which proves its own *in*consistency.

[1] Note that this is different from exams before January 2014.

Recursion Theory

C1. Let X be the set of all e such that W_e is an initial segment of the natural numbers (i.e. W_e is empty, ω or $\{0, 1, .., n\}$ for some $n \in \omega$). Classify the set X in the arithmetical hierarchy.

C2. Let $\mathbf{a} > 0$ be a c.e. degree. Show that there is a d.c.e. degree $\mathbf{d} < \mathbf{0}'$ such that $\mathbf{a} \vee \mathbf{d} = \mathbf{0}'$. (Recall that a Turing degree is d.c.e. if it contains a set of the form $W \setminus V$ for some c.e. sets W and V.)

C3. Say that a real X is *recognizable* if there is some Turing functional Φ such that for all $Y, Z \in 2^\omega$, if exactly one of Y, Z is equal to X then $\Phi^{Y \oplus Z}(0)$ halts and outputs 0 if $Y = X$ and 1 if $Z = X$.

Show that the recognizable reals are exactly the computable reals.

Sketchy Answers or Hints

E1 ans. Let $n_1 < n_2 < \ldots$ be any sequence of increasing positive integers. Consider the ω^* sum of the ω^{n_k}, i.e.,

$$\ldots + \omega^{n_3} + \omega^{n_2} + \omega^{n_1}$$

Show that these are pairwise nonisomorphic.

E2 ans. Suppose that $\bigcup_{n\in\omega} T_n$ had a finite axiomatization $\{\varphi\}$. Then by compactness, some T_n must prove φ. But then $T_n \vdash T_{n+1}$, contradicting the existence of a model of T_n which does not model T_{n+1}. If L is finite, then there are only finitely many L-structures of any given size. Again by compactness, if $\bigcup_{n\in\omega} T_n$ has no infinite model, then all of its models must have size less than K for some K. But then there are only finitely many models of $\bigcup_{n\in\omega} T_n$, and each of these can be completely described by a single formula. If this were true, then $\bigcup_{n\in\omega} T_n$ would be finitely axiomatizable, which is a contradiction to the above.

E3 ans. Consider $T = PA \cup \{\neg\text{con}(PA)\}$. From $\{\neg\text{con}(PA)\}$ and the fact that $PA \subseteq T$ (which is provable in PA given a straightforward enumeration of T), it is easy to give a proof of $\neg\text{con}(T)$.

C1 ans. X is Π^0_2 complete: it is Π^0_2 because $e \in X$ if and only if $(\forall n)[n \in W_e \Rightarrow (\forall m < n)[m \in W_e]]$. It is complete, because $Inf \leq_m X$, which can be proved by a standard construction.

C2 ans. Given a c.e. set A that is not computable, we build a d.c.e. set D and a c.e. set E so that the requirements below are satisfied:

$$S : K = \Gamma^{A,D}$$

$$N_e : E \neq \Phi^D_e$$

We build Γ as a c.e. set of axioms of the form $(A \upharpoonright a(n)+1, D \upharpoonright d(n)+1, n, i)$, where $i = 0, 1$. We can invalidate older axioms by enumerating $d(n)$ in D if

n enters K. In order to satisfy N_e while preserving S we pick a threshold k, wait until S stops modifying $D \restriction d(k)$. Then start an attack with a witness $x_0 > k$: we wait until $\Phi_e^D(x_0) \downarrow= 0$ and if that happens we would like to restrain $D \restriction \varphi_e(x_0)+1$ and enumerate x_0 in E. The restraint might interfere with the global strategy S. Things would be resolved if $A \restriction a(k)$ changes, because then we would be able to move the activity of S above $\varphi_e(x_0)$. We wait for such a change, meanwhile we set things up for a second attack with a new witness $x_1 > x_0$ by enumerating $d(k)$ in D and moving both $d(k)$ and $a(k)$ to new larger values. If we ever we do get the change in A, we can restore $D \restriction \varphi_e(x_0) + 1$ by extracting $d(k)$ again. We repeat this with $x_1, x_2, \ldots$ until we succeed. We must succeed or else we can argue that A is computable.

C3 ans. We show how to determine whether or not $0 \in X$. This strategy can then be used to determine if $1 \in X$, etc.. Search for a $j \in \{0,1\}$ and a finite set of pairs of strings (σ_i, τ_i) so that $\Phi^{\sigma_i \oplus \tau_i}(0) \downarrow= j$ for each i, $0 \preceq \sigma_i$ and $1 \preceq \tau_i$ for each i, and if $j = 0$, then the open sets $[\tau_i]$ cover $[1]$ and if $j = 1$, then the open sets $[\sigma_i]$ cover $[0]$. Some such j and a finite set must exist: Suppose $0 \in X$, then the σ_i's can be taken to all be initial segments of X. Since every Y in $[1]$ has the property that $\Phi^{X \oplus Y}(0) \downarrow= 0$, compactness of 2^ω lets us find a finite set as needed. Similarly if $1 \in X$. Now, once we have found j and this finite set, we must have $0 \in X$ if and only if $j = 1$: Suppose $0 \in X$ and $j = 0$. Then $X \in [1]$, so there is some σ_i, τ_i so that $X \in [\tau_i]$, but then we see that $\Phi^{\sigma_i * 0^\infty \oplus X}(0) = 0$, contrary to X being recognizable. Similarly, we cannot have $0 \notin X$ and $j = 1$.