Three views of LPO and LLPO

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LPO: Limited Principle of Omniscience

LPO: If $f : \mathbb{N} \to \{0, 1\}$ then 0 is in its range or it isn’t.

Written as an $\forall \exists$ formula:

$$\forall f \exists n \left( n = 0 \lor n = 1 \land (n = 0 \leftrightarrow \exists t (f(t) = 0)) \right)$$

A (the) realizer for LPO:

$$R_{\text{LPO}}(f) = \begin{cases} 
0 & \text{if } 0 \in \text{Range}(f) \\
1 & \text{if } 0 \notin \text{Range}(f)
\end{cases}$$
LLPO: Lesser Limited Principle of Omniscience

LLPO: If \( f : \mathbb{N} \rightarrow \{0, 1\} \) and \( 0 \in \text{Range}(f) \), then the first 0 occurs at an even integer or at an odd integer.

Written as an \( \forall \exists \) formula:

\[
\forall f \exists n \left( \exists t \left( f(t) = 0 \right) \rightarrow n \equiv_{\text{mod} \ 2} \mu t(f(t) = 0) \right)
\]

Values of a realizer:

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
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<tbody>
<tr>
<td>( f(n) )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>...</td>
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\( R_{\text{LLPO}}(f) = 0 \)

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<tr>
<th>n</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(n) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
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\( R_{\text{LLPO}}(g) = 1 \)

<table>
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<tr>
<th>n</th>
<th>0</th>
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<th>2</th>
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<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(n) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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\( R_{\text{LLPO}}(h) = 0 \) or 1
Suppose P and Q are problems.

We say P is Weihrauch reducible to Q (and write $P \leq^W Q$) if there are (partial) computable functionals $\Phi$ and $\Psi$ such that if $p$ is an instance of P, then

- $\Phi(p)$ is an instance of Q and
- for any solution $s$ of $\Phi(p)$, $\Psi(s, p)$ is a solution of $p$.

That is, if $R_Q$ is any realizer of Q, then $\Psi(R_Q(\Phi(p)), p)$ is a realizer for P.

If $\Psi$ does not use the original problem $p$, we say P is strongly Weihrauch reducible to Q.
Weihrauch reductions

If $R_Q$ is any realizer of $Q$, then $\Psi(R_Q(\Phi(p)), p)$ is a realizer for $P$.

- $\Phi$ is a pre-processor, turning $P$ problems into $Q$ problems.
- $\Psi$ is a post-processor, turning $Q$ solutions (with copies of $P$ problems) into $P$ solutions.
- $\Phi$ and $\Psi$ uniformly turn any $Q$ realizer into a $P$ realizer.
Example of a Weihrauch reduction

We will show that LLPO $\leq_W$ LPO, relying on post-processing.

Instructions for $\Phi$:

Input an LLPO problem $p$.
Do nothing and output $p$.

Instructions for $\Psi$:

Input $p$ and $R_{LPO}(\Phi(p))$.

Note: $R_{LPO}(\Phi(p))$ is the LPO solution for $p$.

If $R_{LPO}(p) = 1$ (so $p$ is all 1s)
then output 1 and halt.

If $R_{LPO}(p) = 0$ (so $p$ contains a 0)
then find the first 0 and output the location (mod 2).

$\Psi$ is partial. A false value for $R_{LPO}(\Phi(p))$ may loop.
Example of a Weihrauch reduction

We will show that LLPO $\leq_W$ LPO, relying on pre-processing.

Instructions for $\Phi$:
Input an LLPO problem $p$.
For each $n$, define $q(n)$ by
if there is an $m \leq n$ such that $p(m) = 0$ and the first such $m$ is even, then set $q(n) = 0$, and
set $q(n) = 1$ otherwise.

Instructions for $\Psi$:
Input $p$ and $R_{LPO}(\Phi(p))$.
Output $R_{LPO}(\Phi(p))$.

$\Phi$ and $\Psi$ are total. $\Psi$ doesn’t use $p$, so LLPO $\leq_{SW}$ LPO.
For a Weihrauch problem $P$, the parallelization is denoted by $\hat{P}$.

$\hat{P}$ accepts a sequence of $P$ problems as input, and outputs the sequence of their solutions.

For example, written as an $\forall \exists$ formula, $\hat{\text{LPO}}$ is

$$\forall \langle f_i \rangle \exists \langle n_i \rangle \forall i \left( n_i = 0 \lor n_i = 1 \land (n_i = 0 \leftrightarrow \exists t \left( f_i(t) = 0 \right)) \right)$$

which is a set existence statement.
Reverse mathematics: The base theory

Reverse Mathematics measures the strength of theorems by proving equivalence results over...

The base theory RCA₀:

Variables for natural numbers and sets of natural numbers
Axioms

Arithmetic axioms
(e.g. \( n + 0 = n \) and \( n + (m') = (n + m)' \))

Induction for particularly simple formulas

Recursive comprehension:
If you can compute a set, then it exists.
Reverse mathematics: The big five

Many theorems of mathematics are equivalent to one of four statements (over the base theory $\text{RCA}_0$). [3]

$$\text{RCA}_0 < \text{WKL}_0 < \text{ACA}_0 < \text{ATR}_0 < \Pi_1^1\text{-CA}_0$$

Where does \(\hat{\text{LPO}}\) fit?

**Prop (RCA\(_0\))**: The following are equivalent:

1. $\text{ACA}_0$.
2. $\hat{\text{LPO}}$.

For the reversal, find the range of an arbitrary injection on $\mathbb{N}$.
Reverse mathematics: The big five

Many theorems of mathematics are equivalent to one of four statements (over the base theory RCA₀). [3]

\[ \text{RCA₀} < \text{WKL₀} < \text{ACA₀} < \text{ATR₀} < \Pi^1₁\text{-CA₀} \]

Where does \( \hat{\text{LLPO}} \) fit?

Prop (RCA₀): The following are equivalent:

1. WKL₀.
2. \( \hat{\text{LLPO}} \).

For the reversal, separate the ranges of two injections with disjoint ranges.
There is a computable injection on $\mathbb{N}$ with a range that computes $0'$.

There is a computable $\hat{\text{LPO}}$ problem such that every solution computes $0'$.

There is an $\omega$ model of $\text{WKL}_0$ containing only low sets.

Every computable $\hat{\text{LLPO}}$ problem has a low solution.

Thus, $\hat{\text{LPO}} \not\leq_{_W} \hat{\text{LLPO}}$, and so $\text{LPO} \not\leq_{_W} \text{LLPO}$. 
Higher order reverse mathematics

Work in collaboration with Carl Mummert.

Kohlenbach [5] proposed an extension of the axiom systems of reverse mathematics to all finite types.

In this setting, we can prove equivalences between Skolemized functional existence statements.

As an example,

\[ \text{LPO} : \forall f \exists n (n = 0 \lor n = 1 \land (n = 0 \leftrightarrow \exists t (f(t) = 0))) \]

\[ (\text{LPO}) : \exists R_{\text{LPO}} \forall f (R_{\text{LPO}}(f) = 0 \lor R_{\text{LPO}}(f) = 1 \land (R_{\text{LPO}}(f) = 0 \leftrightarrow \exists t (f(t) = 0))) \]
Higher order RM: The base theory

Kohlenbach’s [5] $\text{RCA}_0^\omega$ includes functionals of higher type, like $f : 2^\mathbb{N} \to \mathbb{N}$ and $g : 2^\mathbb{N} \to 2^\mathbb{N}$.

It includes:

- Restricted induction

- Primitive recursion (on $\mathbb{N}$ with parameters)

- $\lambda$-abstraction

Naïvely, if you can compute a functional, it exists.
(LPO) as an ACA$_0$ analog

(LPO) : $\exists R_{LPO} \forall f (R_{LPO}(f) = 0 \lor R_{LPO}(f) = 1 \land (R_{LPO}(f) = 0 \leftrightarrow \exists t (f(t) = 0)))$

(LPO) is Kohlenbach’s ($\exists^2$). The functional $R_{LPO}$ is Kleene’s $E^2$.

RCA$_0^\omega +$ (LPO) is a conservative extension of ACA$_0$ for $\Pi^1_2$ formulas.[6]

And for (LLPO), there is a surprise:

**Prop:** (RCA$_0^\omega$) The following are equivalent:

1. (LPO)
2. (LLPO)
**RCA_0^\omega \vdash (LLPO) \rightarrow (LPO)**

Proof sketch: Working in RCA_0^\omega, suppose we have \( R_{LLPO} \).

Let \( f = \langle 1, 1, 1, \ldots \rangle \), and suppose \( R_{LLPO}(f) = 0 \).

Define a sequence of inputs:

\[
g_n(m) = \begin{cases} 
1 & \text{if } m \neq 2n + 1 \\
0 & \text{if } m = 2n + 1
\end{cases}
\]

For every \( n \), \( R_{LLPO}(g_n) = 1 \), so

\[
\lim_{n} R_{LLPO}(g_n) = 1 \neq 0 = R_{LLPO}(f) = R_{LLPO}(\lim_n g_n)
\]

and \( R_{LLPO} \) is effectively sequentially discontinuous.

Apply Prop. 3.7 of Kohlenbach [5] to obtain (LPO).

If \( R_{LLPO}(f) = 1 \), revise the definition of \( g_n \).
Grilliot’s trick and Kohlenbach’s proposition

We can replace the use of Prop. 3.7 of Kohlenbach [5] with part of the proof of Lemma 1 of Grilliot [4].

Let \( f \) and \( g_n \) be as before. Define the functional \( J : 2^\mathbb{N} \to 2^\mathbb{N} \) for \( h : \mathbb{N} \to 2 \) and \( j \in \mathbb{N} \) by

\[
J(h)(j) = \begin{cases} 
1 & \text{if } \forall x \leq j \ (h(x) \neq 0), \\
g_i(j) & \text{if } i \leq j \land i = \mu t (h(t) = 0).
\end{cases}
\]

Note that if \( h = f \), then \( J(h) = f \). If \( i \) is the least value such that \( h(i) = 0 \), then \( J(h) = g_i \).

So \( h \) contains a zero if and only if \( R_{\text{LLPO}}(J(h)) \neq R_{\text{LLPO}}(f) \). We can use \( R_{\text{LLPO}} \) and \( J \) to compute \( R_{\text{LPO}} \).
The underlying computability theory

Why does $\text{RCA}_0^\omega \vdash (\text{LLPO}) \rightarrow (\text{LPO})$ when $\text{LPO} \not\leq_W \text{LLPO}$?

The two approaches yield different information.

$LPO \not\leq_W \text{LLPO}$ because there are no fixed computable pre-processing and post-processing functionals that can (uniformly) convert every realizer for $\text{LLPO}$ into a realizer for $\text{LPO}$.

$\text{RCA}_0^\omega \vdash (\text{LLPO}) \rightarrow (\text{LPO})$ because given any realizer for $\text{LLPO}$ we can compute a realizer for $\text{LPO}$. In the proof, when we said “suppose $R_{\text{LLPO}}(f) = 0$” we made a non-uniform choice.
What about the other half of the equivalence?

**Prop:** $\text{RCA}_0^\omega \vdash (\text{LPO}) \rightarrow (\text{LLPO})$.

The idea of the proof:

$\text{RCA}_0^\omega$ can prove the existence of the pre-processing and post-processing functionals in our second proof of $\text{LLPO} \leq_w \text{LPO}$. The implication follows by composition of functionals.

The post-processing functional in the first proof was not total, so that argument does not work in the higher order reverse mathematics setting.
Three questions, one answer.

- Are all Skolemized functional existence statements (SFEs) corresponding to WKL\(_0\) statements equivalent to (LPO) in higher order reverse mathematics?

  No. Kohlenbach [5] noted statements about moduli of uniform continuity that are conservative over WKL\(_0\).

- Which SFEs corresponding to WKL\(_0\) are equivalent to (LPO)? Are the others weaker and equivalent to each other, or is there a low-level higher-order zoo?

- What about SFEs corresponding to \(\Pi^1_1\)–CA\(_0\) (and the Suslin functional) as compared to those corresponding to theorems equivalent to ATR\(_0\)?
References


