Toward deciding the $\forall \exists$-theory of the $\Sigma^0_2$-enumeration degrees

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(joint work with Goh, Ng and M. Soskova)
Most “natural” degree structures $D$ are very complicated partial orders and usually follow this pattern:

- The first-order theory of the partial order $D$ is undecidable. In fact, it is usually as complicated as second-order arithmetic (for global degree structures) or first-order arithmetic (for countable local degree structures).
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Therefore, computability theorists often study “fragments” of the first-order theory, determined by a bound on the quantifier depth of the formulas:

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- The $\forall \exists$-theory of $\mathcal{D}$ can “often” be shown to be decidable (more later).
- The $\exists \forall \exists$-theory of $\mathcal{D}$ can “usually” be shown to be undecidable.
Definition

$A \leq_e B$ if there is an enumeration operator $\Phi$ with $A = \Phi(B)$, i.e., there is a c.e. set $\Phi$ of pairs $(x, F)$ (of numbers $x$ and finite sets $F$) denoting that for all $x$, $x \in A$ iff there is $(x, F) \in \Phi$ with $F \subseteq B$. 
In particular, we will focus on the degree structure $S_e$ of the enumeration degrees of the $\Sigma^0_2$-sets, which coincides with the enumeration degrees $a \leq 0'_e$. They form a densely ordered countable upper semilattice with least element $0_e$ (the degree of the c.e. sets) and greatest element $0'_{e}$ (the degree of $\overline{K}$).
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For \( S_e \), the \( \exists \)-theory is decidable by Lagemann (1972), whereas the \( \exists \forall \exists \)-theory is undecidable by Kent (2006).

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However, the decidability of the $\forall \exists$-theory of $S_e$ remains open.
Deciding the $\forall \exists$-theory of a degree structure $D$ amounts to giving a uniform decision procedure to the following

**Algebraic Problem (for deciding the $\forall \exists$-theory of $D$)**

Given finite partial orders $P$ and $Q_i \supseteq P$ (for $i \leq n$), does every embedding of $P$ into $D$ extend to an embedding of $Q_i$ into $D$ for some $i \leq n$ (where $i$ may depend on the embedding of $P$)?
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Two major subproblems of the $\forall \exists$-theory of $S_e$ are decidable:

**Extension of Embeddings Problem**

Given finite partial orders $P$ and $Q \supseteq P$, does every embedding of $P$ into $S_e$ extend to an embedding of $Q$ into $S_e$?

(Lempp/Slaman/Sorbi 2005: complicated decision procedure)
Deciding the $\forall\exists$-theory of a degree structure $D$ amounts to giving a uniform decision procedure to the following

### Algebraic Problem (for deciding the $\forall\exists$-theory of $D$)

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(Lempp/Slaman/Sorbi 2005: complicated decision procedure)

### Lattice Embeddings Problem

Which finite lattices can be embedded into $S_e$ (preserving join and meet)?

(Lempp/Sorbi 2002: all finite lattices embed)
The main technical obstacles to deciding the $\forall \exists$-theory of $S_e$ showed up first in the following

**Theorem (Ahmad 1989 (cf. Ahmad/Lachlan 1998))**

1. There is an *Ahmad pair* of $\Sigma^0_2$-enumeration degrees $(a, b)$, i.e., there are incomparable degrees $a$ and $b$ such that any degree $v < a$ is $\leq b$. 
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1. There is an *Ahmad pair* of $\Sigma^0_2$-enumeration degrees $(a, b)$, i.e., there are incomparable degrees $a$ and $b$ such that any degree $v < a$ is $\leq b$.

2. There is no *symmetric Ahmad pair* of $\Sigma^0_2$-enumeration degrees, i.e., there are no incomparable degrees $a$ and $b$ such that any degree $v < a$ is $\leq b$, and any degree $w < b$ is $\leq a$. 
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These are examples of $\forall \exists$-statements blocking $P \subset Q_0$ but not both $P \subset Q_0$ and $P \subset Q_1$:
In 2007, Kent emailed me the following “next” two questions arising from Ahmad’s work:

**Technical Questions**

1. Is there an *Ahmad triple* of $\Sigma^0_2$-enumeration degrees, i.e., are there degrees $a$, $b$ and $c$ such that $(a, b)$ and $(b, c)$ form Ahmad pairs?
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So, e.g., 1 is an example of simultaneously blocking $\mathcal{P} \subset Q_0, Q_1, Q_2, Q_3$:

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For many years, I believed the answers to both to be “yes”.

---

Steffen Lempp | AE-theory of the Sigma2-enumeration degrees
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2. But there is a *weak Ahmad triple*, i.e., there are pairwise incomparable $\Sigma^0_2$-enumeration degrees $a$, $b$, and $c$ such that $(a, b)$ and $(a, c)$ do not form Ahmad pairs but any degree $v < a$ is $\leq b$ or $\leq c$.

This has led to some exciting on-going work that I will present in more detail in the remainder of the talk.
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As for the other question:

**Theorem (Kalimullin, Lempp, Ng, Yamaleev, submitted)**

There is no cupping Ahmad pair.

The proof turns out to be a non-uniform finite-injury(!) argument.
Given the difficulty of the overall problem of deciding the $\forall\exists$-theory, we are currently concentrating on the following subproblem:

1-Point Extensions of Antichains

Decide, given a finite antichain $P = \{a_0, \ldots, a_n\}$ and 1-point extensions $Q_S = \{a_0, \ldots, a_n, x_S\}$ and $Q_T = \{a_0, \ldots, a_n, x^T\}$ for some nonempty subsets $S, T \subseteq \{0, \ldots, n\}$ (where $x_S < a_i$ iff $i \in S$; and $x^T > a_i$ iff $i \in T$),
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The subproblem involving only extensions $Q^T$ is trivial: Extendible iff there is a singleton $T$. 
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The subproblem involving only extensions $\mathcal{Q}_T$ is trivial: Extendible iff there is a singleton $T$. We have now found a (complicated) complete characterization for the above subproblem involving only extensions $\mathcal{Q}_S$.
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We have now found a (complicated) complete characterization for the above subproblem involving only extensions $Q_S$.

We have no working conjecture that combines the $Q_S$ and the $Q^T$. 

Steen Lempp

AE-theory of the Sigma2-enumeration degrees
### Main Theorem (Goh, Lempp, Ng, M. Soskova, in preparation)

Fix $n > 0$ and $S \subseteq \mathcal{P}(\{0, \ldots, n\}) - \{\emptyset\}$.

Let $S_0 = \{i \leq n \mid \{i\} \in S\}$, and let $S_1 = \{0, \ldots, n\} - S_0$. 

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<tr>
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Then some embedding of \( \mathcal{P} \) into \( S_e \) cannot be extended to an embedding of \( Q_S \) for any \( S \in S \) ("\( S \) can be blocked") iff (*) holds:

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- \( S_0, S_1 \neq \emptyset \) and there is an assignment \( \nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\} \),
  i.e., a function such that
    - for each \( i \in S_0 \), \( \{i\} \cup \nu(i) \notin S \), and
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Let me first give examples for each of the three clauses of (\( \ast \)):
- \( S_0 = \emptyset \): Make the degrees \( a_i \) pairwise minimal pairs.
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Let me first give examples for each of the three clauses of (*):

1. $S_0 = \emptyset$: Make the degrees $a_i$ pairwise minimal pairs.
2. $\{0, \ldots, n\} \neq \bigcup S$: Fix $j \in \{0, \ldots, n\} - \bigcup S$ and make each $a_k$ (for $k \neq j$) form an Ahmad pair with $a_j$. 

The most difficult condition of (*) concerns the assignment 
\( \nu : S_0 \to \mathcal{P}(S_1) - \{\emptyset\} \) satisfying

- for each \( i \in S_0 \), \( \{i\} \cup \nu(i) \notin S \), and
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Easy example showing the first bullet is needed:

\(a_0\) forms an Ahmad pair with \(a_1\);
so \(S_0 = \{0\}\) and \(\nu : 0 \mapsto \{1\}\), namely, \(S = \{\{0\}\}\) can be blocked, but \(\{\{0\}, \{0, 1\}\}\) cannot.
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Harder example showing the second bullet is needed:

- \( a_0 \) and \( a_1 \) both form an Ahmad pair with \( a_2 \), and \( a_0 \) and \( a_1 \) form a minimal pair; so \( S_0 = \{0, 1\} \) and \( \nu : 0, 1 \mapsto \{2\} \), namely, \( S = \{\{0\}, \{1\}, \{0, 1, 2\}\} \) and even \( S = \{\{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\} \) can be blocked.
The most difficult condition of (\*) concerns the assignment \( \nu : S_0 \rightarrow \mathcal{P}(S_1) - \{\emptyset\} \) satisfying

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\( S = \{\{0\}, \{1\}, \{0, 1\}, \{0, 1, 2\}\} \) can be blocked.
But: Note that the first bullet fails for \( S = \{\{0\}, \{1\}, \{0, 2\}\} \), so this cannot be blocked.
Proof Sketch: “$S$ can be blocked” implies (*)&: Suppose $S_0 \neq \emptyset$ and $\{0, \ldots, n\} = \bigcup S$. 
Proof Sketch: “$S$ can be blocked” implies ($\ast$):
Suppose $S_0 \neq \emptyset$ and $\{0, \ldots, n\} = \bigcup S$. We will use the following

**Theorem**

Suppose $a, b_i$ and $c_{i,j}$ (for $i < m$ and $j < n_i$) are degrees with $a \nleq b_i$ and $b_i \nleq c_{i,j}$ for all $i$ and $j$. 
Proof Sketch: “S can be blocked” implies (⋆):
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Then there is either $v < a$ with $v \nleq b_i$ for all $i$; or for some $i$, there is $w < b_i$ with $w \nleq c_{i,j}$.

The proof is a substantial extension of our “no Ahmad triple” result.
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Suppose $a$, $b_i$ and $c_{i,j}$ (for $i < m$ and $j < n_i$) are degrees with $a \not\leq b_i$ and $b_i \not\leq c_{i,j}$ for all $i$ and $j$.
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Now suppose $P$ embeds via degrees $a_i$. For each $i \in S_0$, fix nonzero $v_i < a_i$ with $v_i \not\leq a_k$ for all $k \in S_0 - \{i\}$, and set $\nu(i) = \{j \in S_1 \mid v_i \leq a_j\}$, so $\{i\} \cup \nu(i) \not\in S$ (and $\nu(i) \neq \emptyset$).
Proof Sketch: “\( S \) can be blocked” implies (*):
Suppose \( S_0 \neq \emptyset \) and \( \{0, \ldots, n\} = \bigcup S \). We will use the following

**Theorem**

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The proof is a substantial extension of our “no Ahmad triple” result.

Now suppose \( \mathcal{P} \) embeds via degrees \( a_i \). For each \( i \in S_0 \), fix nonzero \( v_i < a_i \) with \( v_i \not\leq a_k \) for all \( k \in S_0 - \{i\} \), and set \( \nu(i) = \{j \in S_1 \mid v_i \leq a_j\} \), so \( \{i\} \cup \nu(i) \not\in S \) (and \( \nu(i) \neq \emptyset \)).
On the other hand, for \( F \subseteq S_0 \) with \( |F| > 1 \), set \( v_F = \bigcup_{i \in F} v_i \), and so \( v_F < a_j \) iff \( j \in \bigcap\{\nu(i) \mid i \in F\} \) (and \( \bigcap\{\nu(i) \mid i \in F\} \not\in S \)).
Proof Sketch: “$S$ can be blocked” implies ($\ast$):
Suppose $S_0 \neq \emptyset$ and $\{0, \ldots, n\} = \bigcup S$. We will use the following

**Theorem**

Suppose $a$, $b_i$ and $c_{i,j}$ (for $i < m$ and $j < n_i$) are degrees with $a \nless b_i$ and $b_i \nless c_{i,j}$ for all $i$ and $j$.
Then there is either $v < a$ with $v \nless b_i$ for all $i$; or for some $i$, there is $w < b_i$ with $w \nless c_{i,j}$.

The proof is a substantial extension of our “no Ahmad triple” result.

Now suppose $P$ embeds via degrees $a_i$. For each $i \in S_0$, fix nonzero $v_i < a_i$ with $v_i \nless a_k$ for all $k \in S_0 - \{i\}$, and set $\nu(i) = \{j \in S_1 \mid v_i \leq a_j\}$, so $\{i\} \cup \nu(i) \notin S$ (and $\nu(i) \neq \emptyset$).
On the other hand, for $F \subseteq S_0$ with $|F| > 1$, set $v_F = \bigcup_{i \in F} v_i$, and so $v_F < a_j$ iff $j \in \bigcap\{\nu(i) \mid i \in F\}$ (and $\bigcap\{\nu(i) \mid i \in F\} \notin S$).
So $\nu$ is an assignment as desired.
(*) implies “$S$ can be blocked”: $0''$-argument with requirements:
(*) implies “$\mathcal{S}$ can be blocked”: 0‴-argument with requirements:

$$A_i : X = \Phi(A_i) \rightarrow \forall j \in \nu(i) (X = \Gamma_j(A_j)) \text{ or } \exists \Delta (A_i = \Delta(X)) \ (i \in S_0)$$
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$\mathcal{E}_F : \forall k \in F (Y = \Phi(A_k)) \rightarrow Y = \Lambda(A_i)$

(if $F \in S$ and there is a unique $i \in S_0$ with $F \subseteq \nu(i)$)

$\mathcal{E}_{F,j} : \forall k \in F (Y = \Phi(A_k)) \rightarrow Y = \Lambda(A_j)$

(if $F \in S$ and $F \subseteq \nu(i)$ for at least two $i \in S_0$, and $j \in \bigcap \{\nu(i) \mid F \subseteq \nu(i)\} - F$)
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$M_{i,j} : Y = \Phi(A_i) = \Phi(A_j) \rightarrow Y$ is c.e. (if $i \in S_0$; $j \in S - (\{i\} \cup \nu(i)))$

$M_F : \forall j \in F (Y = \Phi(A_j)) \rightarrow Y$ is c.e.

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$(\text{if } |F| > 1, F \subseteq S_1, \text{ and } F \not\subseteq \nu(i) \text{ for all } i \in S_0)$

$I_{j,k} : A_j \neq \Psi(A_k) \ \text{ (if } j,k \in S_1 \text{ and there is } i \in S_0 \text{ with } j,k \in \nu(i))$

$I_j : A_j \neq W \ \text{ (if } j \in S_1 - \bigcup_{i \in S_0} \nu(i))$
Thanks!