Low levels of the arithmetical hierarchy and computable reductions on $\omega$

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Surveying results of many people.
Ceers ($\Sigma^0_1$ equivalence relations)

A lot of work has been focused on the structure of ceers, including:

- There is a universal degree, which appears naturally: Provable equivalence in PA, isomorphism of finite presentations of groups, word problems of some groups, equivalence relations where the classes are uniformly effectively inseparable.

- Ceers with finitely many classes form an initial segment $\mathcal{I}$.

- There are ceers which are not above $=^{\omega}$ (usually called Id). We call these dark. This is a failure of the analog of Silver’s theorem.

- There are infinitely many ceers which are minimal over $\mathcal{I}$.

- We have some descriptions of when pairs of ceers have (or don’t have) a join or a meet.

- Every degree has a strong minimal cover (some only 1, some countably many)
More ceers facts

- $\omega < \omega$ embeds as an initial segment of the degrees (sending the empty string to $\text{Id}$).
- The degree structure of Ceers interprets $(\mathbb{N}, +, \cdot)$ and so the theory is as complicated as possible. Also, the degree structure of the Light ceers, also the degree structure of the Dark ceers. Also, each of these $/\mathcal{I}$.
- The collection of 1-dimensional ceers $R_X$ for $X \subseteq \omega$ embeds the 1-degrees of (infinite) c.e. sets.

**Definition (The Halting Jump operator on ceers)**

Given a ceer $X$, define $X'$ by $i \ X' \ j$ if and only if $\phi_i(i) \downarrow X \phi_j(j) \downarrow$.

- $X' \geq X$ for all $X$.
- $X' > Y'$ iff $X > Y$.
- $X' \equiv X$ if and only if $X$ is universal.
- $X' \leq A \oplus B$ implies $X' \leq A$ or $X' \leq B$. 
There is a universal co-ceer $\pi$.

The only ceer which is below a co-ceer is Id, and the ones with finitely many classes.

Every co-ceer is light (i.e. above Id).
Everything about ceers relativizes (some care needed: Relativizations include $0'$-reductions).

- There is universal $\Sigma^0_2$-equivalence relation.
- There are dark ones.
- There are the 1-dimensional ones (closed downwards)

We haven’t really considered what the halting jump looks like here. e.g., What are there other fixed points besides the universal ceer degree and the universal $\Sigma^0_2$-degree?

For any $\Delta^0_2$-degree $d$, the complete $d$-ceer is a fixed-point. Are there any others? Is the universal ceer least among the fixed points?

Very little independent investigation here.
Many natural examples of things that correspond to ERs on $2^\omega$ restricted to CE: $=^{ce} \equiv \text{Id}^+ \in \Pi^0_2$, $E^{ce}_{set} \equiv \text{Id}^{++} \in \Pi^0_4$, $E^{ce}_3 \in \Pi^0_4$

**Definition**

For any $E$, let $i E^+ j$ if and only if $[W_i]_E = [W_j]_E$.

**Theorem**

There is NO universal $\Pi^0_n$-equivalence relation.

In fact, for every $\Pi^0_n$-equivalence relation $X$, there is some $\Delta^0_n$-equivalence relation which is not below $X$.

This is a constant foot-gun. The temptation to say that $=^{ce}$ is $\Pi^0_2$-universal is overpowering at times. Resist.
**Theorem**

If $X$ is a $\Pi^0_2$-equivalence relation, then there is some $Y \in \Delta^0_2$ so that $i X j$ iff $Y[i] = Y[j]$.

Now, the Ershov-Hierarchy essentially answers why there can’t be a universal one. Consider the sequence:

$=^{ce}$ formed by letting $Y$ be a universal c.e. set.

Next $=^{d-ce}$ formed by letting $Y$ be a universal d-c.e. set.

$\vdots$

$=^{\alpha-ce}$ formed by letting $Y$ be a universal $\alpha$-c.e. set.

$\vdots$

By looking at where $Y$ sits in the Ershov hierarchy, it’s clear that these are co-final among $\Delta^0_2$-equivalence relations.
Aside on $\equiv\Sigma_n$ and $\dagger$

Relativizing at higher levels, that same hierarchy looks like:

$\equiv\Sigma^0_3 \leq d-\Sigma^0_3 \leq \ldots$.

**Theorem**

$\text{Id}^{\dagger n} \equiv \Sigma^0_{2n-1}$.

**Corollary**

Every $\Sigma^0_{2n-1}$ or $\Pi^0_{2n-1}$ equivalence relation reduces to $\text{Id}^{\dagger n}$.

**Proof.**

If $X$ is $\Sigma^0_{2n-1}$, we provide a reduction of $X$ to $\equiv\Sigma^0_{2n-1}$. Send $n$ to $[n]_X$.

If $X$ is $\Pi^0_{2n-1}$, send $n$ to $\omega \setminus [n]_X$. 

**Question**

Is $\pi^{\dagger} \equiv \Sigma^0_2$?
Aside on $\equiv^{\Sigma_n}$ and $\vdash 2$

$\vdash$ doesn’t preserve these difference hierarchies:

**Question**

For any $\Pi^0_n$-equivalence relation $X$, $X^{\vdash} \leq \equiv^{\Sigma_{n+1}}$.

**Proof.**

Send $i$ to $[W_i]_X$.

**Question**

We can ask about what the high $\Pi^0_n$-equivalence relations are. This has been looked at for the ceers with some surprising answers, but not even at $\Pi^1_0$.

Is $=^{ce}$ the least $\Pi^0_2$-equivalence relation $X$ so that $X^{\vdash} \equiv \equiv^{\Sigma_3}$?

Do they all have that jump?
So why is there a $\Pi^0_1$-universal?

**Theorem**

For every $\Pi^0_1$ relation (not assumed transitive) $E$, there is a $\Delta^0_1$ set $X$ and a partial computable function $f$ so that if $E$ is an equivalence relation, then $i E j$ iff $X[f(i)] = X[f(j)]$.

**Proof.**

At every $s$, we determine $X(\langle n, m \rangle)$ for $n, m \leq s$. Let $t_0 = 0$ and let $t_{n+1}$ be the first stage $> t_n$ where $E$ looks transitive on $[0, n + 1]$. If $E$ is transitive, then this is an infinite sequence of stages, and $f : n \mapsto t_n$ will be our reduction. When $s$ is not a $t_n$-stage for some $n$, we do nothing much in coding $X$ – make no differences. Put 0 on all new inputs.

Otherwise, code the highest-priority split – use transitivity to make all the coding columns look okay.

We could do this for $\Pi^0_2$-relations, but the reduction function $f$ would also be $\Delta^0_2$, so we wouldn’t get computable reduction.
Here lie some natural ERs on c.e. sets:

\[ E_{0}^{ce} \equiv E_{1}^{ce} \equiv E_{2}^{ce} \equiv \text{the } \Sigma_{3}^{0}-\text{universal degree} \]

**Definition**

\[
\begin{align*}
i E_{0}^{ce} j & \text{ iff } W_{i} =* W_{j} \\
i E_{1}^{ce} j & \text{ iff for all but finitely many } n, \ W_{i}[n] = W_{j}[n] \\
i E_{2}^{ce} j & \text{ iff } \Sigma_{n \in A \Delta B} \frac{1}{n} < \infty
\end{align*}
\]

The pattern seems to be that almost any “natural” \( \Sigma_{n}^{0} \)-equivalence relation will collapse to being universal. Obviously, this doesn’t happen at \( \Pi \)-levels.

Some classes within \( \Sigma_{3}^{0} \)-ERs, including the following two attempts to “effectivize” the class of countable borel equivalence relations (cbers).
Definition (Coskey, Hamkins, R. Miller (2012))

- The action of a computable group $G$ acting on $\mathbf{CE}$ is computable in indices if there is computable $\alpha$ so that

$$W_{\alpha(g,e)} = g \cdot W_e.$$ 

The induced orbit equivalence relation is denoted $E^c e_G$.

- $E^c e$ is enumerable in indices if there is computable $\alpha$ so that, for all $i \in \omega$,

$$e E^c e i \iff (\exists n)(W_{\alpha(e,n)} = W_i).$$

The first here was a natural attempt to use the Feldman-Moore theorem to bring the idea of cbers to ERs on $\mathbf{CE}$. The second attempt is similar, but using the Luzin-Novikov theorem.
Theorem

If $G$ is a computable group acting on $\text{CE}$ computably in indices, then either $E_G^{ce} \equiv E_0^{ce}$ or $E_G^{ce} \equiv^{ce}$

First, we showed that any group acting on $\text{CE}$ computably in indices is actually acting via a permutation on $\omega$. Still, there are several computable subgroups of $S_\infty$ to consider. The prototypical examples to consider come down to the following cases:

- Let $G$ be all finite permutations of $\omega$.
- Let $\mathbb{Z}$ act on $\omega$ by shifting.
- Let $G$ be generated by $(0, 1)(2, 3, 4)(5, 6, 7, 8) \cdots$.

Having shown these were all $\Sigma_3^0$-complete, we realized that we had enough tricks to prove the same for any infinite $G \subseteq S_\infty$. 
Theorem
There are infinite chains and antichains of ERs which are enumerable in indices between $=^{ce}$ and $E_0^{ce}$.

Simple construction for chains.
For $X \subset \omega$, let $F(X)$ be the least element in $X^c$.
Let $iR_n j$ if and only if $W_i = W_j$ or $0 \in W_i \cap W_j$ and $F(W_i) \equiv F(W_j) \mod n$.
Note that $=^{ce}$ reduces to $R_n$ by sending $W_i$ to $W_i + 1$. Among c.e. sets which contain 0, there are $n + 1$ classes depending $F(W_i) \mod n$ OR $F(W_i) = \infty$. The last one is $\Pi_2^0$-complete, while the others are $\Sigma_2^0$-complete. By counting the number of properly $\Sigma_2^0$-classes, you can show $R_{n+1} \nleq R_n$.  

Our examples are all $\Delta^0_3$. Can there be a properly $\Sigma^0_3$, but not universal, ER which is enumerable in indices?

Also, there is a $\Delta^0_2$ enumerable in indices ER: $E_{\text{min}}$, and a $\Pi^0_2$ which is below $\equiv^{ce} E_{\text{max}}$.

Can there be a $\Sigma^0_2$ one which is not $\Delta^0_2$. More generally, can there be any $\Sigma^0_2$ quotient of $\equiv^{ce}$ which is not $\Delta^0_2$?
Can the Lusin-Novikov direction be salvaged by demanding more uniformity from the enumerations?

**Definition**

$E^{ce}$ is *uniformly* enumerable in indices if there is a computable $\alpha$ so that for all $i \in \omega$,

\[
e^{ce} i \iff (\exists n)(W_{\alpha(e,n)} = W_i).
\]

and whenever $W_e = W_i$, $W_{\alpha(e,n)} = W_{\alpha(i,n)}$.

Note that you expect this if the operation $W_i \mapsto W_{\alpha(i,n)}$ is really an operation on sets (i.e., is independent of the enumeration).

**Observation**

$E^{ce}$ is uniformly enumerable in indices if and only if it is the orbit equivalence of a computable action of a monoid $M$ on $CE$. 
Thank you

for your attention, comments and contributions!