Minimal covers in the Weihrauch degrees

Steffen Lempp

University of Wisconsin-Madison

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(joint work with J. Miller, Pauly, M. Soskova and Valenti)
A mathematical problem can be viewed as a statement of the form

$$\forall X (\varphi(X) \rightarrow \exists Y \psi(X, Y)),$$

where $\varphi$ and $\psi$ are formulas in the (two-sorted) language $\mathcal{L} = \{+, \cdot, <, 0, 1, \in\}$ using only number quantifiers.

Here, $X$ is called an instance, and $Y$ a solution of the problem.

Two standard examples are:

- **Weak König’s Lemma**: $X$ is an infinite binary tree by $\varphi(X)$, and $Y$ is an infinite path through $X$ by $\psi(X, Y)$;

- **Ramsey’s Theorem for Pairs and 2 Colors**: $X$ is a 2-coloring of unordered pairs of numbers by $\varphi(X)$, and $Y$ is an infinite homogeneous set by $\psi(X, Y)$.

We consider mathematical problems from three angles: the proof-theoretic, the model-theoretic and the computability-theoretic one.
The proof-theoretic angle: *Reverse Mathematics*

We work over a weak base theory, usually $\text{RCA}_0$ ($\text{PA}^-$ with $\Sigma^0_1$-Induction and $\Delta^0_1$-Comprehension, essentially codifying computable mathematics), and measure the proof-theoretic strength of mathematical problems in the usual proof calculus.

E.g., one can show that Weak König’s Lemma and Ramsey’s Theorem for Pairs and 2 colors are independent over $\text{RCA}_0$. Ramsey’s Theorem for Pairs with 2 colors and with 3 colors are equivalent, but strictly weaker than Ramsey’s Theorem for Triples with 2 colors.

On the one hand, this approach is less restrictive: We can use assumptions repeatedly. But our proof (thinking model-theoretically, i.e., semantically) has to work for any model of arithmetic, including non-standard models, which may not satisfy full (first-order) induction. (E.g., the Infinite Pigeonhole Principle does not follow from $\text{RCA}_0$.)
The model-theoretic angle: \( P \leq_\omega Q \)

Instead of considering all models of our theory, we can only consider models with a standard first-order part (so-called \( \omega \)-models, with an (often countable) second-order part \( S \subseteq \mathcal{P}(\omega) \)).

We then work with semantic implication: A problem \( P \) is reducible to a problem \( Q \) if every model \((\omega, S)\) of \( Q \) is a model of \( P \).

This approach has not been explored very much. (It is sometimes called the \( \omega \)-model reducibility and denoted as \( Q \models_\omega P \).) It avoids “pesky” problems with induction.

E.g., the Infinite Pigeonhole Principle is just outright true (in \( \omega \)-models of \( \text{RCA}_0 \)).
The (less restrictive) computability-theoretic approach:

Call \( P \) computably reducible to \( Q \) \((P \leq_c Q)\) if

- every \( P \)-instance \( X \) computes a \( Q \)-instance \( \hat{X} \), and
- every \( Q \)-solution \( \hat{Y} \) to this \( \hat{X} \), together with \( X \), computes a \( P \)-solution \( Y \) to \( X \).

This approach is more restrictive: We can use assumptions only once but can argue computability-theoretically.

(If \( Y \) can be computed only from \( \hat{Y} \) without using \( X \), we write \( P \leq_{sc} Q \).)

E.g., now Ramsey’s Theorem for Pairs with 3 colors does not computably reduce to Ramsey’s Theorem for Pairs with 2 colors.
The (more restrictive) computability-theoretic approach: \textit{Weihrauch reducibility}

We restrict the previous approach by requiring uniformity: $P \leq_{W} Q$ if there are Turing functionals $\Phi$ and $\Psi$ (the \textit{forward} and the \textit{backward} functionals) such that

- every $P$-instance $X$ computes a $Q$-instance $\hat{X} = \Phi(X)$, and
- every $Q$-solution $\hat{Y}$ to this $\hat{X}$, together with $X$, uniformly computes a $P$-solution $Y = \Psi(\hat{Y} \oplus X)$ to $X$.

This is the most restrictive approach: We are allowed to query $Q$ only once, and only uniformly so.

(If $Y$ can be computed only from $\hat{Y}$ as $\Psi(\hat{Y})$, we write $P \leq_{sW} Q$.)

E.g., we have $\text{DNR}_2 \leq_{c} \text{DNR}_3$ but $\text{DNR}_2 \not\leq_{W} \text{DNR}_3$. 
Much research about Weihrauch reducibility concerns applications, often via “representing” problems in other spaces via “names” in $\mathbb{N}^\mathbb{N}$.

However, we consider the *Weihrauch degrees* as a *degree structure*: So we change notation:

Consider problems as partial multi-valued functions $f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}$, mapping problems $x$ satisfying $\varphi(x)$ to the set of all solutions $y$ satisfying $\psi(x, y)$.

We denote the set of partial multi-valued functions $f : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}$ by $\mathcal{P}\mathcal{F}$, and the quotient $(\mathcal{P}\mathcal{F}/ \equiv_{\mathcal{W}}, \leq)$ (with the induced partial order) by $\mathcal{W}$. 
Basic Facts about $\mathcal{W}$:

$\mathcal{W}$ is a partial order with least element $0 = \{\emptyset\}$. Under AC, $\mathcal{W}$ has no greatest (or even maximal) element.

$\mathcal{W}$ has size $2^c = 2^{2^{\aleph_0}}$. In fact, every Weihrauch degree $\neq 0$ has size $2^c$.

Every nontrivial lower cone in $\mathcal{W}$ has size $2^c$.

Every nontrivial maximal antichain in $\mathcal{W}$ must be uncountable. There is a maximal antichain of size $2^c$, but nothing more is known.

Every well-ordered ascending chain in $\mathcal{W}$ of countable cofinality has an upper bound.

For every $\kappa \leq c$ of uncountable cofinality, there is an ascending chain in $\mathcal{W}$ of type $\kappa$ without upper bound. (This is open for $c < \kappa \leq 2^c$.)
Quite a few natural operations on $\mathcal{PF}$ have been defined, some of which are degree-theoretic, and some of which are not.

The following operations of meet and join make $\mathcal{W}$ into a distributive lattice:

$$f \sqcup g : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}, \quad (f \sqcup g)(i, x) = \begin{cases} 
\{0\} \times f(x), & \text{if } i = 0, \\
\{1\} \times g(x), & \text{if } i = 1;
\end{cases}$$

$$f \sqcap g : \subseteq \mathbb{N}^\mathbb{N} \Rightarrow \mathbb{N}^\mathbb{N}, \quad (f \sqcap g)(x, y) = (\{0\} \times f(x)) \cup (\{1\} \times g(y)).$$

The next “natural” degree-theoretic question concerns the (un)decidability and complexity of the first-order theory of $\mathcal{W}$.

The Weihrauch degree $1 = \deg(id)$ of the identity function

$$\text{id} : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}, x \mapsto x$$

plays a special role as we will now explore.
The Lattice of the Medvedev Degrees:

A *mass problem* is a subset $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$.

A mass problem $\mathcal{A}$ is *Medvedev reducible* to a mass problem $\mathcal{B}$ ($\mathcal{A} \leq_M \mathcal{B}$) if there is a Turing functional $\Phi$ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$. (So, in particular, $\Phi(x)$ is a total function for all $x \in \mathcal{B}$.)

Denote the quotient $(\mathcal{P}(\mathbb{N}^\mathbb{N})/ \equiv_M, \leq)$ of *Medvedev degrees* by $\mathcal{M}$.

We next define, for each $\mathcal{A} \subseteq \mathbb{N}^\mathbb{N}$, the function $d_\mathcal{A} :\subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ mapping each $x \in \mathcal{A}$ to $0^\omega$. (Note $d_\mathcal{A} \equiv_W \text{id} \upharpoonright \mathcal{A}$.)

Then the map $d : \mathcal{P}(\mathbb{N}^\mathbb{N}) \rightarrow \mathcal{P}\mathcal{F}$, $\mathcal{A} \mapsto d_\mathcal{A}$ induces an embedding of $\mathcal{M}^{op}$ (the Medvedev degrees under the reverse ordering) into $\mathcal{W}$ (by Higuchi/Kihara 2013, following Brattka/Gherardi 2011). This embedding is *onto* the cone $\mathcal{W}(\leq 1)$ in the Weihrauch degrees below $\deg_W(\text{id})$.

So note $\mathcal{M}^{op} \cong \mathcal{W}(\leq 1) = \{\deg_W(\text{id} \upharpoonright \mathcal{A}) \mid \mathcal{A} \subseteq \mathcal{P}(\mathbb{N}^\mathbb{N})\}$. 
Question (Pauly 2020)
Is \(1 = \text{deg}_W(id)\) definable in \((W, \leq)\)?:

Theorem (Lempp, J. Miller, Pauly, M. Soskova, Valenti)
The degree \(1\) is definable in \((W, \leq)\) in two ways:
1. \(1\) is the greatest degree that is a strong minimal cover in \(W\).
2. \(1\) is the least degree such that the cone above it is dense.

Theorem (Lewis-Pye, Nies, Sorbi 2009, Shafer 2011)
The first-order theory of \((M, \leq)\) is as complicated as third-order arithmetic.

Corollary (Lempp, J. Miller, Pauly, M. Soskova, Valenti)
The first-order theory of \((W, \leq)\) (and of \((W(\leq 1), \leq))\) is as complicated as third-order arithmetic.
Proof Sketch (1):

For $x \in \mathbb{N}^\mathbb{N}$, let $\{x\}^+ = \{(e)^\exists y \mid \Phi_e(y) = x \text{ and } y \not\leq_T x\}$.

**Theorem (Dyment 1976)**

In the lattice of the Medvedev degrees $(\mathcal{M}, \leq, \wedge, \vee)$:

- $\mathcal{B}$ is a minimal cover of $\mathcal{A}$ iff there is $x \in \mathcal{A}$ with $\mathcal{A} \equiv \mathcal{M} \mathcal{B} \wedge \{x\}$ and $\mathcal{B} \wedge \{x\}^+ \equiv \mathcal{M} \mathcal{B}$.

- The strong minimal covers are precisely of the form $(\deg_M(\{x\}), \deg_M(\{x\}^+))$ for any $x \in \mathbb{N}^\mathbb{N}$.

So being the Medvedev degree of a singleton (i.e., being a degree of solvability) is definable in $\mathcal{M}$.

**Corollary**

$1 = \deg_W(id)$ is a strong minimal cover of $\deg_W(id | \text{NREC})$, where $\text{NREC} = \deg_M(\{0^\omega\}^+) = \deg_M(\{x \in \mathbb{N}^\mathbb{N} \mid x >_T 0^\omega\})$. 
Theorem (Lempp, J. Miller, Pauly, M. Soskova, Valenti)

In the Weihrauch degrees $(\mathcal{W}, \leq)$:

- $\text{deg}_W(g)$ is a *minimal cover* of $\text{deg}_W(f)$ iff $g \equiv_W f \uplus id|\{x\}$ for some $x \in \mathbb{N}^\mathbb{N}$ with $\text{dom}(f) \not\leq_M \{x\}$ and $\text{dom}(f) \leq_M \{x\}^+$.

- $\text{deg}_W(g)$ is a *strong minimal cover* of $\text{deg}_W(f)$ iff there is $x \in \mathbb{N}^\mathbb{N}$ with $g \equiv_W id|\{x\}$ and $f \equiv_W id|\{x\}^+$.

In particular, $\text{deg}_W(id)$ is the greatest strong minimal cover in $\mathcal{W}$, and every Weihrauch degree has at most one strong minimal cover.

Our proof critically relies on the following

Lemma

- If $\text{deg}_W(g)$ is a minimal cover of $\text{deg}_W(f)$, then there is $h$ with $|\text{dom}(h)| = 1$ such that $g \equiv_W f \uplus h$.

- If $\text{deg}_W(g)$ is a strong minimal cover of $\text{deg}_W(f)$, then there is $h \equiv_W g$ with $|\text{dom}(h)| = 1$. 
Proof of “If $\text{deg}_W(g)$ is a minimal cover of $\text{deg}_W(f)$, then there is $h$ with $|\text{dom}(h)| = 1$ such that $g \equiv_W f \sqcup h$.”

Construct $\xi : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ as $\xi = \bigcup_{s \in \omega} \xi_s$ for finite functions $\xi_s$.

Define $G_\xi(x, \xi(x)) = g(x)$.

Then $G_\xi \leq_W g$ for all $\xi$. We try to ensure $f <_W f \sqcup G_\xi <_W g$ by letting $\xi$ “scramble” the domain of $g$.

At odd stages, we try to ensure $G_\xi \not<_W f$ via the pair $(\Phi_e, \Phi_i)$.
At even stages we try to ensure $g \not<_W f \sqcup G_\xi$ via the pair $(\Phi_e, \Phi_i)$.

So this construction has to start failing at some finite stage $s$ with some $\xi_s$.
This gives $g \equiv_W f \sqcup G_{\xi_s}$ for a finite function $G_{\xi_s}$.
But $G_{\xi_s} \equiv_W \bigsqcup_{i_n} h_i$ for functions $h_i$ with $|\text{dom}(h_i)| = 1$.
Since $\text{deg}_W(g)$ is a minimal cover of $\text{deg}_W(f)$, we have $g \equiv_W f \sqcup h_j$ for some $j < n$. 
Proof Sketch (2):

We rely on the following

**Lemma**

The following are equivalent for $f \in \mathcal{P}F$:

- $\text{id} \not\leq_W f$;
- There are $g, h \in \mathcal{P}F$ such that $f \leq_W g <_W h$ and $\deg_W(h)$ is a minimal cover of $\deg_W(g)$.

Thus, in particular, the Weihrauch degrees $\geq 1$ are dense, and $1$ is least such.
Thank you!