WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

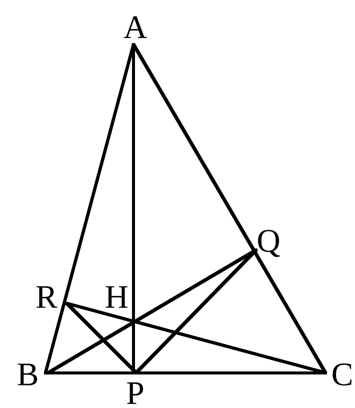
SOLUTIONS TO PROBLEM SET IV (2001-2002)

1. It is known that $(1+\sqrt{2})^{99}$ is not an integer. Nevertheless, show that if we write this number in its decimal representation, then there are at least 25 consecutive 0's directly following the decimal point.

SOLUTION. Write the polynomial $(1+x)^{99}$ as $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{99}x^{99}$, where all the coefficients a_i are nonnegative integers. If we substitute $-x$ for x, we obtain $(1-x)^{99} = a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots - a_{99}x^{99}$, and it follows that $(1+x)^{99} + (1-x)^{99}$ is a sum of integer multiples of powers of x with *even* exponents. Now if $x = \sqrt{2}$, then $x^{2m} = 2^m$ is an integer, and therefore $(1+\sqrt{2})^{99} + (1-\sqrt{2})^{99}$ is some nonnegative integer that we will call N. We can thus write $(1+\sqrt{2})^{99} = N + d$, where $d = -(1-\sqrt{2})^{99} = (\sqrt{2}-1)^{99}$, since 99 is odd. Also $d > 0$, since $\sqrt{2} - 1 > 0$. By using a calculator, we see that $d = (\sqrt{2}-1)^{99}$ is approximately equal to 1.274×10^{-38}, so d can be written as $0.000\cdots001274\cdots$, where there are exactly 37 zeros between the decimal point and the digit 1. Since N is an integer, it follows that $(1+\sqrt{2})^{99} = N+d$ has exactly 37 consecutive zeros to the right of the decimal point.

Even without a calculator, it is easy to see that at least 29 consecutive zeros occur. Indeed, since $\sqrt{2} < 1.5$, we have $d < (.5)^{99} = (1/2)^{99} = 2/2^{100}$. But $2^{10} = 1024 > 10^3$, so $2^{100} > 10^{30}$, and we conclude that $d < 2 \times 10^{-30}$, which is the number $0.000\cdots002$, where there are 29 zeros between the decimal point and the digit 2.

2. Altitudes $\overline{AP}$, $\overline{BQ}$ and $\overline{CR}$ are drawn in $\triangle ABC$, and these lines meet at point H, as indicated. (Recall that the three altitudes of a triangle always go through a common point, which is called the orthocenter of the triangle.) Suppose that $AH = BC$. Show that $\overline{PR}$ and $\overline{PQ}$ are perpendicular.



SOLUTION. Since $\triangle APC$ is a right triangle, we see that $\angle HAQ$ is complementary to $\angle ACB$. Similarly, working in the right $\triangle BQC$, we see that $\angle CBQ$ is also complementary to $\angle ACB$, and we conclude that $\angle HAQ = \angle CBQ$. Also $\angle AQH = 90^\circ = \angle BQC$ and we are given that $AH = BC$, so it follows that $\triangle AHQ \cong \triangle BCQ$ by the SAA congruence criterion. Thus $BQ = AQ$, so $\triangle AQB$ is an isosceles right triangle and we have $\angle RBH = 45^\circ$. Now consider the four points B, R, H and P. Since $\angle BRH = 90^\circ$, it follows that point R lies on the unique circle that has $\overline{BH}$ as a diameter, and similarly, point P also lies on this circle. Thus, since $\angle RBH$ and $\angle RPH$ are inscribed in this circle and subtend the same arc, it follows that $\angle RPH = \angle RBH = 45^\circ$. Exactly similar reasoning shows that $\angle QPH = 45^\circ$, and thus $\angle RPQ = 90^\circ$, as required.

3. Find all positive integers c, if any, such that the equation $(m^2+1)(n^2+1) = (cmn+1)^2+1$ has infinitely many positive integer solutions m, n.

SOLUTION. Suppose $(m^2+1)(n^2+1) = (cmn+1)^2+1$ has a solution with both m and n positive integers. Then $2m^2 \geq m^2+1$ and $2n^2 \geq n^2+1$, so

$$(2m^2)(2n^2) \geq (m^2+1)(n^2+1) = (cmn+1)^2+1 > c^2m^2n^2.$$

Thus $c^2 < 4$ and, since c is a positive integer, we must have $c = 1$. In other words, $c = 1$ is the only possibility, and it remains to see whether the equation $(m^2+1)(n^2+1) = (cmn+1)^2 + 1$ with $c = 1$ can have infinitely many solutions. To this end, note that

$$(m^2+1)(n^2+1) - (mn+1)^2 - 1 = m^2 - 2mn + n^2 - 1 = (m-n)^2 - 1.$$

Thus we get infinitely many solutions by letting m, n be any pair of consecutive positive integers.

4. (New Year's Problem). Let a denote the average of the reciprocals of the numbers

$$\sqrt{10^6+n+1} + \sqrt{10^6+n}$$

with $n = 0, 1, 2, \ldots, 2000$. Show that a can be written as a fraction u/v, where u and v are positive integers, and find the smallest possible value for $u + v$.

SOLUTION. If s denotes the sum of the reciprocals of the numbers $\sqrt{10^6+n+1} + \sqrt{10^6+n}$ with $n = 0, 1, 2, \ldots, 2000$, then the average a of these numbers is equal to $s/2001$. To compute s, first note that

$$\begin{aligned}\left(\sqrt{10^6+n+1} + \sqrt{10^6+n}\right)\cdot\left(\sqrt{10^6+n+1} - \sqrt{10^6+n}\right)&\\ = (10^6+n+1) - (10^6+n) = 1.&\end{aligned}$$

Thus, the reciprocal of $\sqrt{10^6+n+1} + \sqrt{10^6+n}$ is equal to $\sqrt{10^6+n+1} - \sqrt{10^6+n}$, and therefore

$$\begin{aligned}s = {}&\left(-\sqrt{10^6} + \sqrt{10^6+1}\right) + \left(-\sqrt{10^6+1} + \sqrt{10^6+2}\right)\\ &+ \left(-\sqrt{10^6+2} + \sqrt{10^6+3}\right) + \cdots + \left(-\sqrt{10^6+2000} + \sqrt{10^6+2001}\right).\end{aligned}$$

Since all the middle terms in this sum cancel in pairs, it follows that

$$s = -\sqrt{10^6} + \sqrt{10^6+2001} = -1000 + 1001 = 1.$$

Hence $a = s/2001 = 1/2001 = u/v$, and since $1/2001$ is in lowest terms, it is clear that the smallest possible value for $u + v$ is $1 + 2001 = 2002$.

5. If x is an integer, then certainly $x^2 + x$ and $x^3 + 2x^2$ are integers. If x is rational, but not an integer, then it is easy to see that neither $x^2 + x$ nor $x^3 + 2x^2$ is an integer. Do there exist nonrational real numbers x so that both $x^2 + x$ and $x^3 + 2x^2$ are integers? If so, find all possibilities for x.

SOLUTION. Let x be a nonrational real number with $x^2 + x$ and $x^3 + 2x^2$ both equal to integers, say $x^2 + x = a$ and $x^3 + 2x^2 = b$. Then x is a root of the quadratic polynomial $X^2 + X - a$ and the cubic polynomial $X^3 + 2X^2 - b$. In addition, x must be a root of the remainder when we divide $X^3 + 2X^2 - b$ by $X^2 + X - a$. Now, polynomial long division yields

$$X^3 + 2X^2 - b = (X+1)(X^2 + X - a) + (a-1)X + (a-b),$$

so x is a root of $(a-1)X + (a-b)$. It follows that if $a \neq 1$, then $x = (b-a)/(a-1)$ is rational, a contradiction. Thus we must have $a = 1$ and then $b = a = 1$. In particular, x is a root of $X^2 + X - 1$, so the quadratic formula implies that $x = (-1 \pm \sqrt{5})/2$. Conversely, if x is either of these two values, then $x^2 + x - 1 = 0$, so $x^2 + x = 1$ is an integer. Furthermore, $x^3 + 2x^2 - 1 = (x+1)(x^2 + x - 1) = 0$, so $x^3 + 2x^2 = 1$ is also an integer.