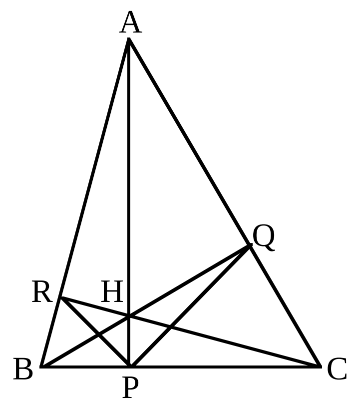
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET IV (2001-2002) **JANUARY 2002**



1. It is known that $(1+\sqrt{2})^{99}$ is not an integer. Nevertheless, show that if we write this number in its decimal representation, then there are at least 25 consecutive 0's directly following the decimal point.

2. Altitudes $\overline{AP}$, $\overline{BQ}$ and $\overline{CR}$ are drawn in $\triangle ABC$, and these lines meet at point H, as indicated. (Recall that the three altitudes of a triangle always go through a common point, which is called the orthocenter of the triangle.) Suppose that $AH = BC$. Show that $\overline{PR}$ and $\overline{PQ}$ are perpendicular.

3. Find all positive integers c, if any, such that the equation $(m^2+1)(n^2+1) = (cmn+1)^2+1$ has infinitely many positive integer solutions m, n.

4. (New Year's Problem). Let a denote the average of the reciprocals of the numbers

$$\sqrt{10^6+n+1}+\sqrt{10^6+n}$$

with $n = 0, 1, 2, \ldots, 2000$. Show that a can be written as a fraction u/v, where u and v are positive integers, and find the smallest possible value for $u+v$.

5. If x is an integer, then certainly x^2+x and x^3+2x^2 are integers. If x is rational, but not an integer, then it is easy to see that neither x^2+x nor x^3+2x^2 is an integer. Do there exist nonrational real numbers x so that both x^2+x and x^3+2x^2 are integers? If so, find all possibilities for x.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the page. Remember that solutions usually require a proof or justification.

RETURN TO: MATHEMATICS TALENT SEARCH
Dept. of Mathematics, 480 Lincoln Drive
University of Wisconsin, Madison, WI 53706

DEADLINE
February 11
2002

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(Please Detach)

Last Name First Name Grade

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PROBLEM	SCORE
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PROBLEM SET IV