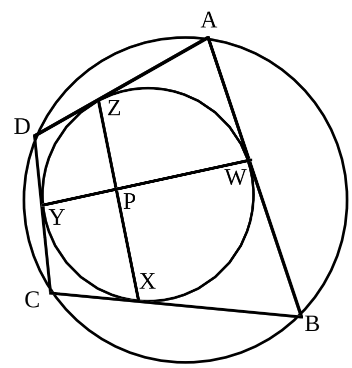
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2001-2002)

1. Find five consecutive positive integers such that the first one is divisible by 5, the next one by 7, the third by 9, the fourth by 11, and the fifth one by 13.

SOLUTION. The numbers 5, 7, 9, 11, 13 almost work except that they are spaced 2 apart. If we could divide them all by 2, they would be spaced 1 apart, but unfortunately they would no longer be integers. Now suppose that N is an odd positive integer divisible by 5, 7, 9, 11 and 13. Then $N+5$, $N+7$, $N+9$, $N+11$, $N+13$ are clearly even integers spaced 2 apart and having the appropriate divisibility property. Thus $(N+5)/2$, $(N+7)/2$, $(N+9)/2$, $(N+11)/2$, $(N+13)/2$ will solve the problem. The smallest choice for N is of course $N = 5{\cdot}7{\cdot}9{\cdot}11{\cdot}13 = 45,045$, and then the first number in the consecutive sequence is $(N+5)/2 = 22,525$.

2. Quadrilateral $ABCD$ is inscribed in a circle and also has a circle inscribed in it. If W, X, Y and Z are the points of tangency of the small circle with the sides of the quadrilateral, as shown, prove that $\overline{WY}$ and $\overline{XZ}$ are perpendicular.



SOLUTION. Since $\overline{AZ}$ and $\overline{AW}$ are tangent to the smaller circle, we have $\angle A = (\overset{\frown}{ZY} + \overset{\frown}{YX} + \overset{\frown}{XW} - \overset{\frown}{WZ})/2$. Similarly, since $\overline{CY}$ and $\overline{CX}$ are tangent lines from C, we have $\angle C = (\overset{\frown}{ZY} + \overset{\frown}{WZ} + \overset{\frown}{XW} - \overset{\frown}{YX})/2$. Thus $\angle A + \angle C = \overset{\frown}{ZY} + \overset{\frown}{XW} = 2\angle ZPY$, where P is the indicated point of intersection of $\overline{WY}$ and $\overline{XZ}$. Finally, since $\angle A$ and $\angle C$ are inscribed in the larger circle, it follows that $\angle A + \angle C = (\overset{\frown}{BCD} + \overset{\frown}{BAD})/2 = 360^\circ/2 = 180^\circ$. Thus $2\angle ZPY = 180^\circ$, so $\angle ZPY = 90^\circ$ and consequently $\overline{WY}$ and $\overline{XZ}$ are perpendicular.

3. Let the sequence of integers $a_1, a_2, a_3, \ldots$ be defined by $a_1 = 2$, $a_2 = 3$, $a_3 = 7$, $a_4 = 43$, and in general $a_{n+1} = 1 + a_1 a_2 \cdots a_n$ for all $n \geq 1$. If $s_n = 1/a_1 + 1/a_2 + \cdots + 1/a_n$ and $p_n = 1/(a_1 a_2 \cdots a_n)$ are the sum and the product of the reciprocals of the first n numbers in the sequence, compute $s_{100} + p_{100}$.

SOLUTION. For convenience, write $t_n = s_n + p_n$. Then $t_1 = s_1 + p_1 = 1/2 + 1/2 = 1$, $t_2 = 5/6 + 1/6 = 1$, $t_3 = 41/42 + 1/42 = 1$ and $t_4 = 1805/1806 + 1/1806 = 1$. This leads us to suspect that $t_n = 1$ for all n, and in particular that all t_n are equal. To prove the latter, note that $s_{n+1} = s_n + 1/a_{n+1}$ and $p_{n+1} = p_n/a_{n+1}$. Thus $p_n - p_{n+1} = p_n(1 - 1/a_{n+1}) = p_n(a_{n+1} - 1)/a_{n+1}$. But $a_{n+1} - 1 = a_1 a_2 \cdots a_n = 1/p_n$, so $p_n(a_{n+1} - 1)/a_{n+1} = 1/a_{n+1}$ and $p_n - p_{n+1} = 1/a_{n+1} = s_{n+1} - s_n$. In other words, $t_n = s_n + p_n = s_{n+1} + p_{n+1} = t_{n+1}$ and this clearly implies that all t_n are equal. In any case, it follows that $t_1 = t_2 = \cdots = t_{99} = t_{100}$, so $s_{100} + p_{100} = t_{100} = t_1 = 1$.

4. Does there exist a binary operation $\Box$ defined on the set of positive real numbers R^+ such that $(x \Box x) \Box (x \Box x)$ is never equal to $((x \Box x) \Box x) \Box x$ even though $(x \Box y) \Box x = x \Box (y \Box x)$ for all $x, y \in R^+$. Recall that $\Box$ is a binary operation on R^+ means that, for all x and y in R^+, $x \Box y$ is a positive real number determined by x and y.

SOLUTION. There exist many such examples, and we construct some of these based on addition in R^+. Let us fix a positive real number α and define $x \Box y = \alpha(x + y)$, so that $\Box$ is certainly a binary operation on R^+. Furthermore, it is easy to check directly that the weak associative law $(x \Box y) \Box x = x \Box (y \Box x)$ holds. In fact, since it is clear that the commutative law $x \Box y = y \Box x$ is satisfied, we get $(x \Box y) \Box x = x \Box (x \Box y) = x \Box (y \Box x)$. Now we consider the two 4-term expressions $(x \Box x) \Box (x \Box x)$ and $((x \Box x) \Box x) \Box x$. Notice that if $\alpha = 1$, then $x \Box y = x + y$ is ordinary addition and hence each of these 4-term expressions is equal to $4x$. On the other hand, if $\alpha = 1/2$, then $x \Box y$ is the average of x and y, so $x \Box x = x$ for all x. It then follows that $(x \Box x) \Box (x \Box x) = x \Box x = x$ and $((x \Box x) \Box x) \Box x = (x \Box x) \Box x = x \Box x = x$. Thus, for our example, we cannot allow either $\alpha = 1$ or $\alpha = 1/2$. However, as we will see below, any other $\alpha > 0$ will suffice. Indeed, note that $x \Box x = 2\alpha x$, so $(x \Box x) \Box (x \Box x) = (2\alpha x) \Box (2\alpha x) = (2\alpha)^2 x$. Furthermore, $((x \Box x) \Box x) \Box x = ((2\alpha x) \Box x) \Box x = (\alpha(2\alpha + 1)x) \Box x = \alpha(2\alpha^2 + \alpha + 1)x$. In particular, if $(x \Box x) \Box (x \Box x) = ((x \Box x) \Box x) \Box x$, then $4\alpha^2 x = \alpha(2\alpha^2 + \alpha + 1)x$ and, since $x \neq 0$ and $\alpha \neq 0$, it follows that $4\alpha = 2\alpha^2 + \alpha + 1$. Thus $0 = 2\alpha^2 - 3\alpha + 1 = (\alpha - 1)(2\alpha - 1)$ and $\alpha = 1$ or $1/2$. In other words, any positive α different from 1 or $1/2$ yields the required example.

5. Let $f_1, f_2, f_3, \ldots$ be a sequence of integers satisfying $f_{n-1} + f_n = 2n$ for all $n \geq 2$. If $f_1 = 100$, find f_{1000}.

SOLUTION. If we know f_{n-1}, then the formula $f_{n-1} + f_n = 2n$ determines f_n. Thus, for example, when $n = 2$ we have $f_1 = 100$ and hence $f_2 = 2 \cdot 2 - f_1 = -96$. Next, we apply the formula with $n = 3$ to obtain $f_3 = 2 \cdot 3 - f_2 = 102$. By repeating this process, we can certainly determine f_{1000} in 997 more steps. However, we avoid this tedious computation and, in fact, find a precise mathematical expression for f_n. To start with, we have $f_{n-1} + f_n = 2n$ and $f_{n-2} + f_{n-1} = 2(n-1)$ for all $n \geq 3$. Thus, by subtracting the second formula from the first, we obtain $f_n - f_{n-2} = 2n - 2(n-1) = 2$. In other words, $f_n = 2 + f_{n-2}$ for all $n \geq 3$. But then $f_{n-2} = 2 + f_{n-4}$, so $f_n = 2 + f_{n-2} = 2 + (2 + f_{n-4}) = 4 + f_{n-4}$. Continuing this procedure clearly yields

$$f_n = 2 + f_{n-2} = 4 + f_{n-4} = 6 + f_{n-6} = 8 + f_{n-8} = \cdots$$

where we end with either $(n-2) + f_2$ if n is even or $(n-1) + f_1$ if n is odd. Thus we have $f_n = (n-2) - 96 = n - 98$ if n is even and $f_n = (n-1) + 100 = n + 99$ if n is odd. In particular, the answer to the problem is $f_{1000} = 1000 - 98 = 902$.