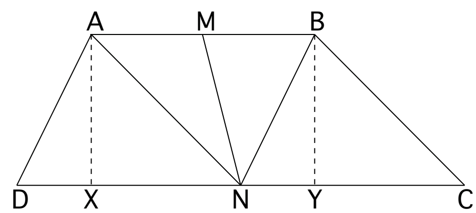
**WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH**

**SOLUTIONS TO PROBLEM SET II (2009-2010)**

**1.** Show that the equation $a^2+b^2=c^2+5$ has infinitely many different positive integer solutions.

**SOLUTION.** We can find an integer solution to the equation $a^2-5=c^2-b^2$ whenever $a$ is odd. Writing $a=2n+1$, we have $a^2-5=4n^2-4n-4$, so we want $(c-b)(c+b)=c^2-b^2=4n^2-4n-4$. We can take $c-b=2$ and $c+b=2n^2-2n-2$, but we must check that we can solve these two equations with $b$ and $c$ being integers. Adding the equations yields $2c=2n^2-2n$, so $c=n^2-n$, and we have $b=c-2=n^2-n-2$. For each integer $n$, therefore, $a=2n+1$, $b=n^2-n-2$ and $c=n^2-n$ yields a solution to the equation $a^2+b^2=c^2+5$.

**2.** In the figure, $ABCD$ is a quadrilateral, $M$ and $N$ are the midpoints of sides $\overline{AB}$ and $\overline{CD}$, respectively, and line $\overline{MN}$ cuts the quadrilateral into two parts with equal areas. Show that sides $\overline{AB}$ and $\overline{CD}$ are parallel.




**SOLUTION.** Draw lines $\overline{AN}$ and $\overline{BN}$ as shown, and consider $\triangle AMN$ and $\triangle BMN$. Viewing $\overline{AM}$ and $\overline{BM}$ as bases for these triangles, we see that these bases are equal. The triangles also have equal heights, and thus they have equal areas.

By assumption, quadrilaterals $AMND$ and $BMNC$ have equal areas, so $\triangle DAN$ and $\triangle CBN$ have equal areas. These triangles have equal bases $\overline{DN}$ and $\overline{CN}$, and so they must have equal heights, and it follows that the perpendicular distances from points $A$ and $B$ to line $\overline{DC}$ are equal. In other words. if we drop perpendiculars $\overline{AX}$ and $\overline{BY}$ from $A$ and $B$ to $\overline{DC}$, we have $AX=BY$. Also, $AX$ and $BY$ are parallel, so $ABXY$ is a parallelogram. (In fact, it is a rectangle.) It follows that $\overline{AB}$ is parallel to $\overline{XY}$, and hence to $\overline{CD}$, as wanted.

**3.** Black and White are playing the following game using the number line as a game board. First, Black places some markers on positive integer points. Then White starts at the origin and repeatedly jumps to the right, always landing at an integer, and always moving at least 1 unit. The rules specify that White's first jump is at most 100 units, and that each later jump is no longer than the previous one. (Thus, for example, White's first four jumps might be 100, 95, 95 and 30 units.) Black wins if White lands on one of Black's markers, and White wins if he can get beyond all of the markers without landing on any of them. Find the smallest number of Black markers that will guarantee a Black win.

**SOLUTION.** Black can guarantee a win with 50 markers but not with fewer. First, we show that Black wins if he has markers at all integer points from 51 to 100, inclusive, and thus 50 markers suffice. Certainly Black wins if White lands on a marker on the first jump, and by the rules of the game, White cannot jump past point 100 on the first jump, so we can assume that White's first jump lands on some number $k \le 50$. Since, all of White's later jumps have length at most $k$, and thus no more than 50, it is impossible for White to jump over the 50 consecutive Black markers.

Now to see that White can win if Black places fewer than 50 markers, consider the following 50 White strategies corresponding to the integers $r$ with $0 \le r \le 49$. Strategy $r$ is this: jump $50+r$ units on the first jump and 50 units on each successive jump. If $P_r$ is the set of points that White will land on following strategy $r$, then $P_r$ is exactly the set of numbers of the form $r+50+50m$ for all integers $m \geq 0$. Thus the numbers in $P_r$ all leave the remainder $r$ when divided by 50. Since such remainders $0 \leq r \leq 49$ are unique, it follows that the 50 paths $P_r$ are all disjoint. Thus if Black places fewer than 50 markers, at least one of White's strategies will miss all of them.

**4.** Let $x$, $y$ and $z$ be integers, and consider the quantity $Q = 16(x^2+y^2+z^2) - 5(x+y+z)^2$. Prove that $Q \geq 0$, and find the smallest positive number that $Q$ can be.

**SOLUTION.** Since $Q = 11x^2 + 11y^2 + 11z^2 - 10xy - 10xz - 10yz$, we can check that

$$Q = x^2 + y^2 + z^2 + 5[(x-y)^2 + (y-z)^2 + (z-x)^2] \,.$$

Now $Q$ is a sum of squares of integers, and because none of these squares can be negative, it follows $Q \geq 0$. Indeed, if two of $x$, $y$ and $z$ are different, then $(x-y)^2+(y-z)^2+(z-x)^2 \geq 1$ and $Q \geq 5$. On the other hand, if $x = y = z$, then $Q = 3x^2$ and, since $x$ is an integer, the smallest positive value this takes on is 3, and this happens when the common value of $x$, $y$ and $z$ is $\pm 1$.

**5.** Let

$$S_n = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{4n+1} - \frac{1}{4n+3} \,.$$

Show that $S_n < 5/6$ for every integer $n \geq 0$.

**SOLUTION.** Observe that $1/a - 1/(a+2) = 2/(a(a+2))$ for all $a$, and thus

$$S_n = \frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \cdots + \frac{2}{(4n+1)(4n+3)} \,.$$

Now let

$$M_n = \frac{2}{3 \cdot 5} + \frac{2}{7 \cdot 9} + \frac{2}{11 \cdot 13} + \cdots + \frac{2}{(4n-1)(4n+1)} \,.$$

Then

$$\begin{aligned} S_n + M_n &= \frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \cdots + \frac{2}{(4n+1)(4n+3)} \\ &= \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{4n+1} - \frac{1}{4n+3} \,. \end{aligned}$$

Since all but the first and last terms of the latter sum cancel, we deduce that $S_n + M_n < 1$. Next, observe that $S_n - 2/3$ is a sum of fractions with the property that each of them is smaller than the corresponding term of $M_n$. (For example, $2/(5 \cdot 7) < 2/(3 \cdot 5)$ and $2/(9 \cdot 11) < 2/(7 \cdot 9)$ and so on.) It follows that $M_n > S_n - 2/3$, and we have $2S_n - 2/3 < S_n + M_n < 1$. This yields $S_n < 5/6$, as required. It can be shown that $S_n < \pi/4$ for all integers $n$, and that in fact, $\pi/4$ is the smallest upper bound that works for all $n$.