WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET III (2009-2010)

1. Find a simple formula for the sum $2{\cdot}4 + 3{\cdot}8 + 4{\cdot}16 + 5{\cdot}32 + \cdots + n{\cdot}2^n$, and prove that your formula is correct.

SOLUTION. Let us write S_n to denote the sum of the $n-1$ numbers in the given list. For $n = 2, 3, 4, 5, 6$, we compute that $S_n = 8, 32, 96, 256, 640$, and we observe that these sums can be written as $1{\cdot}2^3, 2{\cdot}2^4, 3{\cdot}2^5, 4{\cdot}2^6, 5{\cdot}2^7$. This suggests that the general formula should be $S_n = (n-1)2^{n+1}$. This formula can be proved by induction, but perhaps the following is a more elegant proof. Starting with the definition of S_n and doubling it, we have

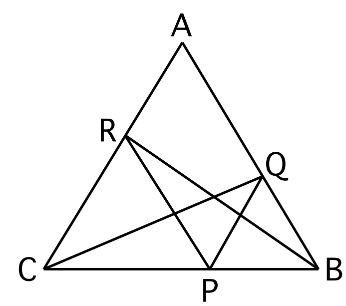
$$S_n = 2{\cdot}4 + 3{\cdot}8 + 4{\cdot}16 + 5{\cdot}32 + \cdots + \qquad n{\cdot}2^n \qquad \text{and}$$
$$2S_n = \qquad 2{\cdot}8 + 3{\cdot}16 + 4{\cdot}32 + \cdots + (n-1){\cdot}2^n + n{\cdot}2^{n+1}\,.$$

Subtracting the second equation from the first, we get

$$-S_n = 2{\cdot}4 + 8 + 16 + 32 + \cdots + 2^n - n{\cdot}2^{n+1}\,.$$

It is easy to see that $4 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1}$, and this yields $-S_n = 2^{n+1} - n{\cdot}2^{n+1}$, and thus $S_n = (n-1)2^{n+1}$, as wanted.

2. In the diagram, P is a point on side $\overline{BC}$ of equilateral $\triangle ABC$. Lines $\overline{PQ}$ and $\overline{PR}$ are drawn parallel to $\overline{AC}$ and $\overline{AB}$, respectively, where Q lies on $\overline{AB}$ and R lies on $\overline{AC}$, and then $\overline{BR}$ and $\overline{CQ}$ are drawn. Prove that $BR = CQ$.



SOLUTION. Observe that $\triangle BQP$ is equilateral since all of its angles are 60°, and thus $QP = BP$. Similarly, $\triangle PRC$ is equilateral, and we have $RP = PC$. Also, we see that $\angle BPR = 180^\circ - \angle RPC = 120^\circ$, and $\angle QPC = 180^\circ - \angle QPB = 120^\circ$. It follows that $\triangle RPB \cong \triangle CPQ$ by side-angle-side, and we conclude that $BR = CQ$ as wanted, because $\overline{BR}$ and $\overline{CQ}$ are corresponding parts of these congruent triangles.

3. Let $n \ge 1$ be an integer and let t denote the number of positive integer divisors of n^2. Show that the number of positive integer solutions (a, b) of the equation $1/a - 1/b = 1/n$ is precisely equal to $(t-1)/2$.

SOLUTION. Given an arbitrary positive integer n, we count the number of positive integer solutions for the equation $1/a - 1/b = 1/n$. Given a solution (a, b), we have $1/a = 1/n + 1/b > 1/n$, so $a < n$ and we can write $a = n - x$ for some integer x, where $0 < x < n$. Then $1/b = 1/a - 1/n = 1/(n-x) - 1/n = x/(n(n-x))$, and we have $b = n(n-x)/x = n^2/x - n$. To count solutions, therefore, we must count positive integers $x < n$ such that the quantity n^2/x is an integer. In other words, we must count positive divisors of n^2 that are less than n.

Now we are given that t is the total numbers of divisors of n^2. One of these t divisors is n, and the remaining $t-1$ divisors can be paired by combining the divisor d with the divisor n^2/d. Exactly one member of each of these $(t-1)/2$ pairs is less than n, so the number of positive divisors of n^2 that are less than n is $(t-1)/2$, where t is the total number of positive divisors of n^2. It follows that the number of positive solutions to the equation $1/a - 1/b = 1/n$ is exactly $(t-1)/2$, as required.

4. (The new year's problem.) Find the smallest positive integer n with the property that the equation $1/a - 1/b = 1/n$ has exactly 2010 different solutions in positive integers a and b.

SOLUTION. From the preceding problem, if t denotes the number of positive integer divisors of n^2, then we know that the number of positive integer solutions of the equation $1/a - 1/b = 1/n$ is precisely $(t-1)/2$.

Now to compute t, write $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where the p_i are different prime numbers and the exponents e_i are positive integers. Then $n^2 = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r}$, and the positive divisors of n^2 are the numbers m of the form $m = p_1^{f_1} p_2^{f_2} \cdots p_r^{f_r}$, where each exponent f_i satisfies $0 \le f_i \le 2e_i$. In other words, there are $2e_i + 1$ possibilities for the exponent f_i, and we conclude that $t = (2e_1+1)(2e_2+1)(2e_3+1)\cdots(2e_r+1)$.

In the given problem, we are told that there are exactly 2010 positive solutions to the equation $1/a - 1/b = 1/n$, and thus $(t-1)/2 = 2010$, and we have $t = 4021$. Now 4021 happens to be a prime number, and so the equation $4021 = t = (2e_1+1)(2e_2+1)(2e_3+1)\cdots(2e_r+1)$ forces the number r of factors to be 1, and $(2e_1+1) = 4021$. Then $e_1 = 2010$, and the number n must have the form $n = p^{2010}$, where p is a prime number. Also, this analysis shows that for every number of the form p^{2010}, the number of solutions is 2010. Since $p = 2$ is the smallest prime, the smallest positive integer n such that the equation $1/a - 1/b = 1/n$ has exactly 2010 positive solutions is therefore 2^{2010}.

5. For any two integers x and y, we write $x \,\square\, y$ to denote a certain integer that is determined by x and y. Suppose that the "$\square$" operation satisfies the following axioms.

(a) $(x \,\square\, y) + (y \,\square\, z) + (z \,\square\, x) = 0$ for all x, y, z.

(b) $z(x \,\square\, y) = (zx) \,\square\, (zy)$ for all x, y, z.

(c) There exist integers x, y with $x > y$ and such that $x \,\square\, y = 1$.

Compute $2010 \,\square\, 10$.

SOLUTION. Setting $x = y = z$, Axiom (a) yields $3(x \,\square\, x) = 0$, and thus $x \,\square\, x = 0$ for all x. Now setting $z = y$ in Axiom (a) yields $(x \,\square\, y) + (y \,\square\, y) + (y \,\square\, x) = 0$, and since the middle term equals 0, we conclude that $y \,\square\, x = -(x \,\square\, y)$ for all x and y. Now set $x = 1$ and $y = 0$ in Axiom (b) to get $z(1 \,\square\, 0) = z \,\square\, 0$ for all z, and write $k = 1 \,\square\, 0$. Then k is an integer and $z \,\square\, 0 = kz$ for all z. Now set $z = 0$ in Axiom (a), yielding $0 = (x \,\square\, y) + (y \,\square\, 0) + (0 \,\square\, x) = (x \,\square\, y) + ky - kx$, and thus $x \,\square\, y = kx - ky = k(x-y)$ for all x and y. By Axiom (c), we can choose $x > y$ such that $x \,\square\, y = 1$. This yields $1 = k(x-y)$, and hence $k > 0$. Furthermore, since k and $(x-y)$ are integers, we deduce that $k = 1$. We now have $x \,\square\, y = k(x-y) = x - y$ for all x and y, and thus $2010 \,\square\, 10 = 2000$.