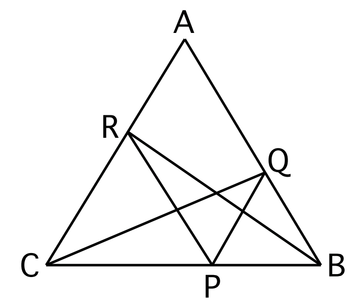
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET III (2009-2010) **DECEMBER 2009**

1. Find a simple formula for the sum $2{\cdot}4 + 3{\cdot}8 + 4{\cdot}16 + 5{\cdot}32 + \cdots + n{\cdot}2^n$, and prove that your formula is correct.

2. In the diagram, P is a point on side $\overline{BC}$ of equilateral $\triangle ABC$. Lines $\overline{PQ}$ and $\overline{PR}$ are drawn parallel to $\overline{AC}$ and $\overline{AB}$, respectively, where Q lies on $\overline{AB}$ and R lies on $\overline{AC}$, and then $\overline{BR}$ and $\overline{CQ}$ are drawn. Prove that $BR = CQ$.



3. Let $n \geq 1$ be an integer and let t denote the number of positive integer divisors of n^2. Show that the number of positive integer solutions (a, b) of the equation $1/a - 1/b = 1/n$ is precisely equal to $(t-1)/2$.

4. (The new year's problem.) Find the smallest positive integer n with the property that the equation $1/a - 1/b = 1/n$ has exactly 2010 different solutions in positive integers a and b.

5. For any two integers x and y, we write $x \square y$ to denote a certain integer that is determined by x and y. Suppose that the "$\square$" operation satisfies the following axioms.

(a) $(x \square y) + (y \square z) + (z \square x) = 0$ for all x, y, z.

(b) $z(x \square y) = (zx) \square (zy)$ for all x, y, z.

(c) There exist integers x, y with $x > y$ and such that $x \square y = 1$.

Compute $2010 \square 10$.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on this problem page. Remember that solutions usually require a proof or justification.

Return To **Or Email To**	**MATHEMATICS TALENT SEARCH** Dept. of Mathematics, 480 Lincoln Drive University of Wisconsin, Madison, WI 53706 talent@math.wisc.edu
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Deadline January 4, 2010	
Problem	**Score**
1	
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