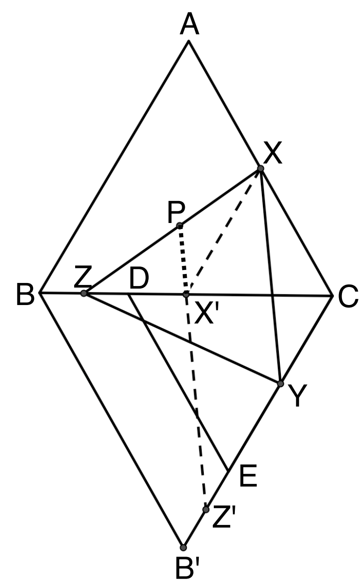
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET V (2009-2010)

1. Suppose that the integers 1 through 9 are written on a blackboard. Erase two of these numbers, which we call a and b, and replace them with the single number $a^2 + 7ab^2 - 3b$. Now there are eight numbers on the board, and we repeat the process: choose two of the eight numbers, call them a and b, and replace them with the one number $a^2 + 7ab^2 - 3b$. Continue like this until just one number remains on the board. Decide whether or not that last remaining number can be 2010.

SOLUTION. It cannot happen that the last number is 2010, and in fact, the last number can never be an even integer. To see why this is so, suppose that the last survivor is even. Since there were some odd numbers at the start, there must have been some moment when the last odd number was erased. If we call the two numbers that were erased at that stage a and b, we know that at least one of a or b is odd and that the number $a^2 + 7ab^2 - 3b$ that replaced a and b is even. It is easy to check, however, that the only way that $a^2 + 7ab^2 - 3b$ can be even is if a and b are both even, and this is a contradiction.

2. In the diagram, $\triangle ABC$ is equilateral and point D lies on side $\overline{BC}$. Also, $\triangle CDE$ is equilateral. If points X, Y and Z are the midpoints of $\overline{AC}$, $\overline{CE}$ and $\overline{BD}$ respectively, show that $\triangle XYZ$ is equilateral.



SOLUTION. Rotate $\triangle ABC$ counterclockwise $60°$ around point C as shown, where the image B' of B under this transformation lies on line $\overline{CE}$, and where X' and Z' are the images of X and Z, and they lie on $\overline{CB}$ and $\overline{CB'}$ respectively. Since X is the midpoint of $\overline{CA}$, we see that X' is the midpoint of $\overline{CB}$, and thus $\overline{XX'}$ is parallel to $\overline{AB}$ and $XX' = (1/2)AB$. Also, $\overline{CB'}$ is parallel to $\overline{AB}$ since $\angle ABC = 60° = \angle BCB'$. Furthermore, $CB' = CB = AB$.

Next, observe that Y is the midpoint of $\overline{CE}$ and Z' is the midpoint of $\overline{EB'}$, and thus $YZ' = (1/2)CB' = (1/2)AB = XX'$. Since also $\overline{YZ'}$ is parallel to $\overline{AB}$, which, in turn, is parallel to $\overline{XX'}$, it follows that a pair of opposite sides of quadrilateral $XYZ'X'$ are both equal and parallel, and thus $XYZ'X'$ is a parallelogram. We conclude that $XY = X'Z'$, and since $X'Z' = XZ$, we see that $XY = XZ$, and so $\triangle XYZ$ is isosceles, and it suffices to show that $\angle ZXY = 60°$. Now, line $\overline{X'Z'}$ meets line $\overline{XZ}$ at a $60°$ angle (at point P) because the rotation was through $60°$. Also $\overline{X'Z'}$ is parallel to $\overline{XY}$ since $XYZ'X'$ is a parallelogram, and it follows that $\overline{XY}$ meets $\overline{XZ}$ at a $60°$ angle, and this completes the proof.

3. There are 15 committees in an organization, and each committee has nine members. One day, there are a total of 96 people present at committee meetings, and none of these committee meetings is attended by anyone who is not a member. Prove that a majority of the committees have a majority of their members present.

SOLUTION. Suppose that there are eight (or more) of the 15 committees that fail to have majorities present at their meetings. Each of these eight committees thus has at most four members present, so there are at most 32 of the 96 people attending these eight sparsely attended meetings.

Then $96 - 32 = 64$ people must be attending the other seven meetings. But each of those seven committees has nine members, so even if every member of every one of those committees is present, that would account for only 63 of the 64 people. This is a contradiction, and we conclude that there are not as many as eight committees that fail to have a majority of the members present. There are thus at most seven such committees, which means that at least eight of the committee meetings are attended by majorities, and that is a majority of the committees.

4. Let w, x, y and z be nonnegative, and suppose that $w + x + y + z = 4$. Show that

$$w^2 + x^3 + y^4 + z^5 \geq w + x^2 + y^3 + z^4.$$

SOLUTION. We want to show that

$$(w^2 + x^3 + y^4 + z^5) - (w + x^2 + y^3 + z^4) \geq 0\,.$$

The left side of this inequality equals $w(w-1) + x^2(x-1) + y^3(y-1) + z^4(z-1)$, and we want to show that this quantity, which we call Q, is nonnegative. For this, observe that

$$z^4(z-1) - (z-1) = (z^4 - 1)(z-1) \geq 0$$

since $z - 1$ is positive if and only if $z^4 - 1$ is positive. Thus, $z^4(z-1) \geq z - 1$, and similarly we have $y^3(y-1) \geq y - 1$, $x^2(x-1) \geq x - 1$ and $w(w-1) \geq w - 1$. We conclude therefore that $Q \geq (w-1) + (x-1) + (y-1) + (z-1) = 0$, as required.

5. Find the largest positive integer n such that the set $\{1, 2, 3, \ldots, n\}$ can be written as the union of two sets X and Y such that neither X nor Y contains the average of two of its members.

SOLUTION. The answer is $n = 8$. We show first that the set $\{1, 2, \ldots, 8\}$ can be written in the form $X \cup Y$, where neither X nor Y contains the average of two of its members. To see this, take $X = \{1, 2, 5, 6\}$ and $Y = \{3, 4, 7, 8\}$.

Next, suppose that $n \geq 9$, and that $\{1, 2, \ldots, n\} = X \cup Y$. We must show that at least one of the sets X or Y contains a number that is the average of two members of that set. Since $n \geq 9$, at least one of X or Y contains at least five of the numbers from 1 to 9. It is no loss to assume that this set is X, so that we can find numbers x_i in X such that $1 \leq x_1 < x_2 < x_3 < x_4 < x_5 \leq 9$. Consider the gaps $g_1 = x_2 - x_1$, $g_2 = x_3 - x_2$, $g_3 = x_4 - x_3$ and $g_4 = x_5 - x_4$. If any of these gaps is 4 or larger, then there are three consecutive numbers in the range from 1 to 9 that are missing from X, and thus they lie in Y. In this case, Y contains the average of two of its members, and we are done. We can assume, therefore, that each gap g_i is 1, 2 or 3.

We consider the possibilities for the list (g_1, g_2, g_3, g_4). If $g_1 = g_2$, then x_2 is the average of x_1 and x_3, so we can assume that $g_1 \neq g_2$, and similarly, we can assume that no two consecutive entries in our list are equal. If $g_1 + g_2 = g_3$, then x_3 is the average of x_1 and x_4, so we can assume that $g_1 + g_2 \neq g_3$, and similarly $g_2 + g_3 \neq g_4$. Similar reasoning allows us to assume that $g_1 + g_2 \neq g_3 + g_4$. Furthermore, we observe that $g_1 + g_2 + g_3 + g_4 = x_5 - x_1 \leq 8$.

Assume first that no $g_i = 3$. Then since every two consecutive g_i must be unequal, the only possibilities are $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$, and both of these are eliminated since the sum of the first two entries in each list equals the sum of the last two entries. It is impossible that exactly one of the g_i is 3 since 3 cannot be either preceded or followed by a consecutive 1 and 2 in either order. Finally, if two of the g_i are 3, the other two must be 1 so that the total will not exceed 8. The only possibilities, therefore, are $(1, 3, 1, 3)$ and $(3, 1, 3, 1)$, and these are eliminated since in each, the sum of the first two entries equals the sum of the last two.