

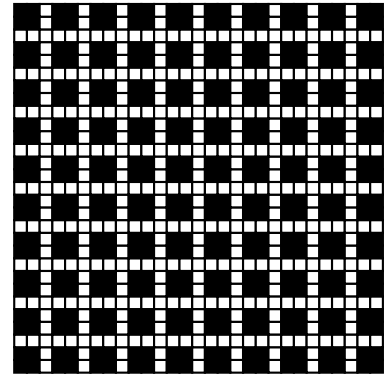
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET II (2012-2013)

1. Prove that a prime number cannot be expressed as the sum of two or more consecutive positive odd integers.

SOLUTION. Consider the consecutive positive odd integers $2a + 1, 2a + 3, \dots, 2(a + b) + 1$ where $0 \leq a$ (so that the numbers are positive) and $1 \leq b$ (so that we have at least two numbers). Then the sum of these numbers is $(b + 1)(2a + b + 1)$. This can be seen by pairing the first number with the last, the second with the second to last etc. and checking that in each pair the sum is $4a + 2b + 2$. (If we have a single number in the middle then it has to be $2a + b + 1$.) Since $0 \leq a$ and $1 \leq b$ both $b + 1$ and $2a + b + 1$ are integers bigger than 1 which means that their product cannot be a prime. This completes the proof of the statement.

2. From a 29×29 grid of unit squares we cut out ninety-nine 2×2 squares consisting the squares of the grid. Show that we can cut out one more!

SOLUTION. Consider the 2×2 square in the up-right corner of the grid, and all copies of this square in the grid which can be obtained by (repeated) shifts of 3 units to the left or down. It is not hard to check that there are exactly 100 such squares in our grid. Let's call these squares special. It is easy to see that if we cut out a 2×2 square then it can only intersect at most one special square (otherwise our 2×2 square would have a square from three consecutive rows or columns which is impossible). But that means that if we cut out 99 squares then at least one of the special squares is still intact and we can cut out one more 2×2 square.

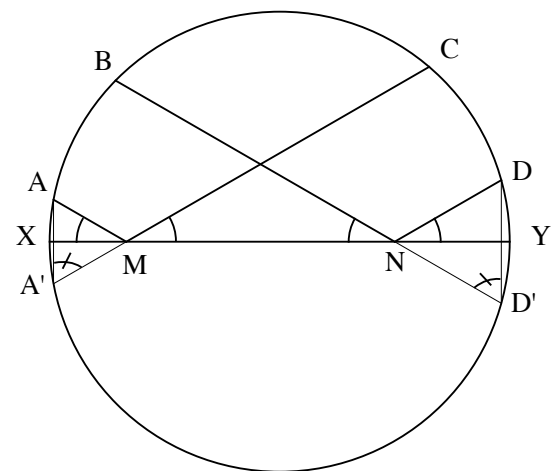


3. A semicircle has a diameter XY on which points M and N lie. The semicircle contains points A, B, C, D such that $\angle AMX = \angle CMY = \angle BNX = \angle DNY$. Prove that $AC = BD$.

SOLUTION.

We consider the whole circle, and extend CM to intersect the circle at A' and BN to intersect the circle at D' . Since $\angle AMX = \angle CMY = \angle XMA'$, we have that $AM = MA'$ by symmetry, and so triangle AMA' is isosceles. Similarly, DND' is isosceles. So $\angle AMA' = 2\angle AMX = 2\angle DNY = \angle DND'$ implies that also $\angle AA'C = \angle AA'M = \angle DD'N = \angle DD'B$, and so arcs AC and BD subtend equal angles and thus $AC = BD$.

(Note that the picture shows one of the many possible configurations, but the proof does not depend on the relative positions of the various points.)



4. We have an infinite sequence of numbers f_1, f_2, f_3, \dots which satisfy

$$f_{\frac{x+y}{3}} = \frac{f_x + f_y}{2}$$

whenever x, y and $\frac{x+y}{3}$ are all positive integers. (f_n denotes the element of the sequence at position n .) How many distinct values can appear in the sequence?

SOLUTION. We will prove that all the numbers must be the same in the sequence. Let x be a positive integer. Then

$$f_x = f_{\frac{x+2x}{3}} = \frac{f_x + f_{2x}}{2}.$$

Solving this for f_{2x} we see that $f_{2x} = f_x$. Applying this repeatedly we get

$$f_{8x} = f_{2 \cdot 4x} = f_{4x} = f_{2 \cdot 2x} = f_{2x} = f_x.$$

Then

$$f_{3x} = f_{\frac{x+8x}{3}} = \frac{f_x + f_{8x}}{2} = \frac{f_x + f_x}{2} = f_x.$$

Setting $x = 1$ the previous identities give $f_1 = f_2 = f_3 = f_4$. We will prove by induction that for any positive integer n we have $f_n = f_1$. We have already proved this for $n = 1, 2, 3, 4$. Now assume that for some positive integer n that $f_n = f_1$, we will show that we also have $f_1 = f_{n+1}$. Indeed,

$$f_{n+1} = f_{\frac{3n+3}{3}} = \frac{f_{3n} + f_3}{2} = \frac{f_n + f_1}{2} = \frac{f_1 + f_1}{2} = f_1$$

which completes the induction. (We used $f_{3n} = f_n$ in the third step.)

5. Show that

$$3 - \frac{1}{5^{2011}} < \underbrace{\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots + \sqrt{6 + \sqrt{6}}}}}}_{2012 \text{ square roots}} < 3.$$

SOLUTION. Let a_n denote the expression with n square roots, e.g. $a_1 = \sqrt{6}$, $a_2 = \sqrt{6 + \sqrt{6}}$. We will show that $3 - \frac{1}{5^{n-1}} < a_n < 3$ for all n . This is certainly true for $n = 1$ as $3 - 1 < \sqrt{6} < 3$. We will proceed by induction, we assume that the inequalities hold for n and we will prove them for $n + 1$. Note that $a_{n+1} = \sqrt{6 + a_n}$ so we need to show

$$3 - \frac{1}{5^n} < \sqrt{6 + a_n} < 3.$$

The second inequality is immediate from $a_n < 3$ by adding 6 to both sides and then taking the square root (which is allowed as the sides are non-negative). The first inequality holds exactly if its square holds: $(3 - 5^{-n})^2 < 6 + a_n$ (again: both sides were non-negative). This is equivalent to

$$a_n > (3 - 5^{-n})^2 - 6 = 3 - 6 \times 5^{-n} + 5^{-2n} = 3 - 5^{-(n-1)} - (5^{-n} - 5^{-2n}).$$

But the last inequality is true since $5^{-n} - 5^{-2n} \geq 0$ and by our assumption $3 - \frac{1}{5^{n-1}} < a_n$. This completes the proof of the first inequality.

Note: The second inequality also follows directly from the simple observation that if we change the last 6 in the expression to 9 then this will increase the value, and in that case the expression can be computed explicitly to be equal to 3.