WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (1996-97)

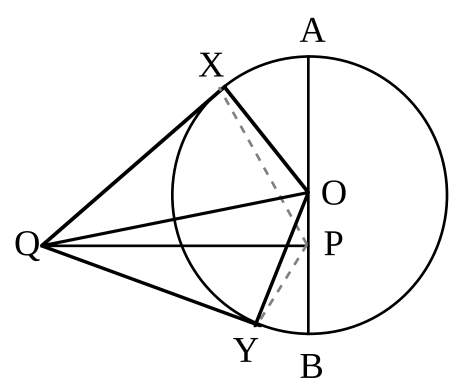
1. If a, b and c are nonzero real numbers, show that $a+b+c$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ cannot both be zero.

SOLUTION. It suffices to show that if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, then $a+b+c\neq 0$. For this, note that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ implies that $ab+ac+bc=abc\big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\big)=0$. Thus

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+ac+bc)=a^2+b^2+c^2>0$$

since a, b and c are nonzero real numbers, and hence $a+b+c\neq 0$.

2. In the figure, $\overline{AB}$ is a diameter of the circle and $\overline{QP}$ is perpendicular to $\overline{AB}$. Also, $\overline{QX}$ and $\overline{QY}$ are the two tangents to the circle from Q. Show that $\overline{QP}$ bisects $\angle XPY$.



SOLUTION. Let O be the center of the given circle and draw line segment $\overline{OQ}$. Note that $\angle OXQ$, $\angle OYQ$ amd $\angle OPQ$ are all right angles, and therefore the points X, Y and P all lie on the circle whose diameter is $\overline{OQ}$. On this circle, arcs $\overset{\frown}{QX}$ and $\overset{\frown}{QY}$ are equal because chords $\overline{QX}$ and $\overline{QY}$ have equal lengths. Since inscribed angles $\angle XPQ$ and $\angle YPQ$ subtend these equal arcs, we deduce that $\angle XPQ=\angle YPQ$, and thus $\overline{QP}$ bisects $\angle XPY$, as required.

3. If x and y are integers (not necessarily positive) satisfying

$$x+x^2+x^8=y+y^2+y^8$$

prove that $x=y$.

SOLUTION. Note that $x^2-y^2=(x-y)(x+y)$ and that

$$\begin{aligned}x^8-y^8&=(x^4-y^4)(x^4+y^4)=(x^2-y^2)(x^2+y^2)(x^4+y^4)\\&=(x-y)(x+y)(x^2+y^2)(x^4+y^4).\end{aligned}$$

Thus, if x and y satisfy $x+x^2+x^8=y+y^2+y^8$, then

$$\begin{aligned}0&=(x-y)+(x^2-y^2)+(x^8-y^8)\\&=(x-y)\big[1+(x+y)+(x+y)(x^2+y^2)(x^4+y^4)\big].\end{aligned}$$

In particular, we see that either $x=y$ or $1+(x+y)+(x+y)(x^2+y^2)(x^4+y^4)=0$. Moreover, in the later case, we have

$$(x+y)\big[1+(x^2+y^2)(x^4+y^4)\big]=-1$$

and hence, if x and y are integers, then $1+(x^2+y^2)(x^4+y^4)$ is a positive integer dividing -1. Thus $1+(x^2+y^2)(x^4+y^4)=1$, so $(x^2+y^2)(x^4+y^4)=0$. This clearly implies that $x=0=y$, and again we have $x=y$.

4. A positive integer $x = d_n d_{n-1} \ldots d_1 d_0$ is said to be *special* if the digits d_i in its decimal expansion satisfy $d_n \le d_{n-1} \le \cdots \le d_1 \le d_0 = 5$. For example, both 15 and 225 are special. Show that there are infinitely many integers x such that both x and x^2 are special.

SOLUTION. Note that both $x=15$ and $x^2=225$ are special. So, perhaps $x=15$ is the start of a sequence of integers x with both x and x^2 are special. To find the next integer in the sequence, we might try $x=115$ or 155 or even 1155, but while these are all special, their squares $x^2=13225$, 24025 and 1334025 are not special.

Next, we observe that $x=35$ is special and so is $x^2=1225$. Also 335 and 3335 are special and so are their squares 112225 and 11122225, respectively. Therefore, we are led to guess that if $x=33\ldots335$, then both x and x^2 are special. This will, of course, guarantee the existence of infinitely many special integers whose squares are also special. To verify our guess, we first note that $10^k-1=\underbrace{99\ldots99}_{k}$, so $\dfrac{10^k-1}{3}=\underbrace{33\ldots33}_{k}$ and $\dfrac{10^k-1}{9}=\underbrace{11\ldots11}_{k}$ for any $k\ge 1$.

Now let x be the special integer

$$x=\underbrace{33\ldots335}_{k}=\underbrace{33\ldots333}_{k}+2=\frac{10^k-1}{3}+2=\frac{10^k+5}{3}$$

for any $k\ge 2$. Then

$$\begin{aligned} x^2 &= \left(\frac{10^k+5}{3}\right)^2=\frac{10^{2k}+2\cdot5\cdot10^k+25}{9}=\frac{10^{2k}+10^{k+1}+25}{9} \\ &= \frac{10^{2k}-1}{9}+\frac{10^{k+1}-1}{9}+3=\underbrace{11\ldots11}_{2k}+\underbrace{11\ldots11}_{k+1}+3 \\ &= \underbrace{11\ldots11}_{k-1}\underbrace{22\ldots22}_{k}5 \end{aligned}$$

since $2k>k+1$. Consequently x^2 is also special.

5. For each integer $k\ge 3$ show that it is possible to write 1 as the sum of the reciprocals of k distinct positive integers.

SOLUTION. We proceed by induction on $k\ge 3$. First, when $k=3$, we have $1=1/2+1/3+1/6$. Now suppose that $k\ge 3$ and that $1=1/n_1+1/n_2+\cdots+1/n_k$ with positive integers $n_1<n_2<\cdots<n_k$. Then $1/2=1/(2n_1)+1/(2n_2)+\cdots+1/(2n_k)$ and therefore

$$1=\frac{1}{2}+\frac{1}{2}=\frac{1}{2}+\frac{1}{2n_1}+\frac{1}{2n_2}+\cdots+\frac{1}{2n_k}.$$

This shows that 1 is the sum of the reciprocals of the k+1 integers 2, $2n_1$, $2n_2$, $\ldots$, $2n_k$. What remains is to show that these numbers are distinct. But clearly $1<n_1$, and therefore we have $2<2n_1<2n_2<\ldots<2n_k$, as required. Thus, the induction step is proved.