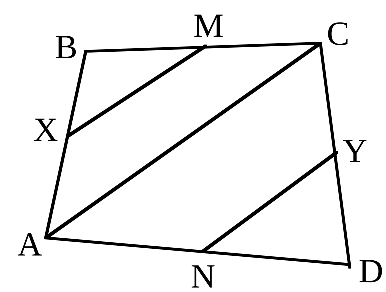
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (1996-97)

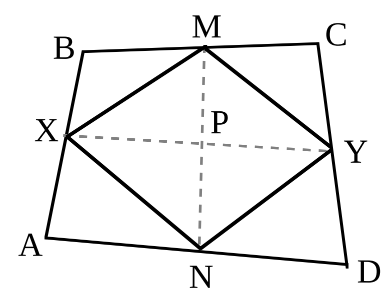
1. Your calculator will tell you that $\sqrt[3]{6\sqrt{3}+10}-\sqrt[3]{6\sqrt{3}-10}$ is approximately equal to 2. Is this quantity exactly equal to 2? Prove that your answer is correct.

SOLUTION. Write $a = \sqrt[3]{6\sqrt{3}+10}$ and $b = \sqrt[3]{6\sqrt{3}-10}$. Then $a^3 - b^3 = 20$ and $ab = \sqrt[3]{108-100} = 2$. Thus $(a-b)^3 = a^3 - b^3 - 3ab(a-b) = 20 - 6(a-b)$ and hence, if $x = a-b$ then $x^3 = 20 - 6x$. Note that $x = 2$ is a solution of this cubic equation and that there are no other real solutions since $x^3 + 6x - 20 = (x-2)(x^2+2x+10)$. It follows that $a - b$ is exactly 2.

2. Let $ABCD$ be a quadrilateral and let X, M, Y and N be respectively the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. Show that the point P where $\overline{XY}$ and $\overline{MN}$ meet is the midpoint of each of $\overline{XY}$ and $\overline{MN}$.



SOLUTION. Draw the lines $\overline{XM}$, $\overline{AC}$ and $\overline{YN}$ as indicated. Since $BM = (1/2)BC$ and $BX = (1/2)BA$, it follows from SAS that $\triangle BXM$ is similar to $\triangle BAC$. In particular, $\angle BXM = \angle BAC$ and hence $\overline{XM}$ and $\overline{AC}$ are parallel. Similarly, $\overline{YN}$ and $\overline{AC}$ are parallel, and consequently $\overline{XM}$ and $\overline{YN}$ are parallel. Now draw the lines $\overline{XN}$ and $\overline{YM}$, and conclude as above that $\overline{XN}$ and $\overline{YM}$ are also parallel. In other words, the quadrilateral $XMYN$ is a parallelogram. Finally, since $\overline{XY}$ and $\overline{MN}$ are the diagonals of this parallelogram, we know that they bisect each other and hence P is the midpoint of each of these lines.



3. (NEW YEAR'S PROBLEM)
Let m and e be positive integers and suppose that $N = 1997m/(m + 1997^e)$ is an integer. Find all possible values for N.

SOLUTION. Note that 1997 is prime. Let's call it p. We first show that the denominator $m + p^e$ is a power of p. To this end, suppose q is any prime divisor of $m + p^e$. Then q must also divide the numerator mp and hence either $q = p$ or q divides m. But in the latter case, we see that q divides $(m + p^e) - m = p^e$, and again we conclude that $q = p$. Thus $m + p^e = p^a$ and clearly $a \geq e + 1$ since $m > 0$. Finally, $m = p^a - p^e$, so

$$N = \frac{mp}{m+p^e} = \frac{(p^a - p^e)p}{p^a} = p - p^{e+1-a}.$$

But N is an integer, so $a \leq e + 1$. Hence $a = e + 1$ and $N = p - p^0 = p - 1 = 1996$.

4. I have a magic money machine into which I can put any number of one dollar coins. If I insert n dollars, the machine returns $2n$ dollars. Each time I use the machine, however, I must insert more money than I did on the previous use. If I start with exactly \$1 and use the machine once, I will have \$2. One my next use of the machine, I am forced to insert \$2 yielding \$4, and on my third use of the machine, I can insert either \$3 or \$4 yielding a total of \$7 or \$8. Consequently, there is no way that I can ever obtain exactly \$3 or \$5 or \$6 by using the machine repeatedly, starting with \$1. Find the largest integer L such that it is impossible to obtain exactly L dollars with the magic money machine, starting with \$1.

SOLUTION. The answer is $L = 10$. Note that \$10 is an impossible amount because after four uses of the machine I must have at least \$11. Now it is easy to check that all amounts from 11 to 20 dollars can be obtained. Suppose, by way of contradiction, that some value larger than \$10 is impossible and let m be the smallest number exceeding 10 such that m dollars cannot be obtained. Then $m > 20$. If m is even, write $m = 2k$ and note that $k > 10$. Thus k dollars is possible and we never inserted as much as k dollars to get it. We can thus insert all k dollars to get $2k = m$ dollars. If m is odd, write $m = 2k + 1$ with $k \geq 10$. Then $k + 1$ dollars is possible and we never inserted as much as k dollars to get it. We can thus insert k dollars yielding a total of $2k + 1 = m$ dollars. This shows that m dollars is possible, contradicting the choice of m. Thus no value larger than \$10 is impossible to obtain.

5. Let S be a subset of the set $\{1, 2, 3, \ldots, 1000\}$ with the property that no sum of two distinct members of S is contained in S. Find the maximum possible number of members in the set S.

SOLUTION. The set $S = \{500, 501, 502, \ldots, 1000\}$ has the desired property and has 501 members. We show now that every set S with this property has at most 501 members. To see this, let a be the smallest member of S and define two more sets of integers by $X = \{s \in S \mid s \neq a\}$ and $Y = \{x + a \mid x \in X\}$. Then X and Y are disjoint, by the property of S, and both are subsets of $T = \{a + 1, a + 2, a + 3, \ldots, a + 1000\}$. Note that T has size $|T| = 1000$. Since $|X| = |Y|$ and $|X| + |Y| \leq |T| = 1000$, we see that $|X| \leq 500$ and thus $|S| = |X| + 1 \leq 501$, as required.