

**WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET II (1997-98)**

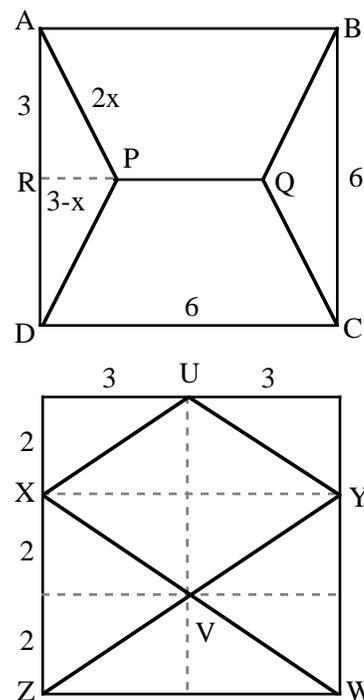
1. Suppose that the quintic equation $x^5 - ax + b = 0$ has two different positive solutions for x . Show that there is also some negative value of x that makes the equation true.

SOLUTION. Suppose that $x = r$ and $x = s$ are two positive solutions to the given equation. Then $r^5 - ar + b = 0 = s^5 - as + b$, and thus $r^5 - s^5 = a(r - s)$. It is easy to check that we can factor $r^5 - s^5 = (r - s)(r^4 + r^3s + r^2s^2 + rs^3 + s^4)$, and this yields $a = r^4 + r^3s + r^2s^2 + rs^3 + s^4$. Now $b = ar - r^5$, and since we have expressed a in terms of r and s , we deduce that $b = r^4s + r^3s^2 + r^2s^3 + rs^4$. We conclude that $b > 0$ since both $r > 0$ and $s > 0$.

Now consider the function $f(x) = x^5 - ax + b$, and note that $f(0) = b > 0$. We will show that there exists some negative number t such that $f(t) < 0$. From this it will follow that the graph of $y = f(x)$ must cross the x -axis at some point between t and 0 , and this yields a negative solution to the equation $f(x) = 0$. (We are using the fact that a polynomial function is “continuous”, which tells us that its graph cannot jump from a negative value to a positive value without actually crossing the x -axis.) To find a negative number t such that $f(t) < 0$, first choose a positive number w large enough so that $w^4 > 2a$ and $w^5 > 2b$. Then $a/w^4 < 1/2$ and $b/w^5 < 1/2$. It follows that $b/w^5 + a/w^4 < 1$, and hence $b + aw < w^5$. If we set $t = -w$, then t is negative and we obtain $b - at < -t^5$. Thus $f(t) = t^5 - at + b < 0$, as desired.

2. We wish to place six points within a 6×6 square so that the minimum distance between any two of the points is as large as possible. For example, we could place the points at A, B, C, D, P and Q in the diagram with $PA = PD = PQ = QB = QC$. Compute the minimum distance in this case and find a configuration in which the minimum distance between points is even larger.

SOLUTION. In the first configuration, the minimum distance is clearly equal to PA , and let us call this length $2x$. To find x , extend the line \overline{PQ} to intersect \overline{AD} at point R . Then $\angle ARP$ is a right angle and $AR = AD/2 = 3$. Also $PQ = 2x$, so $RP = 3 - x$ and the Pythagorean Theorem yields $(2x)^2 = 3^2 + (3 - x)^2 = x^2 - 6x + 18$. Thus $x^2 + 2x - 6 = 0$, so $x = -1 \pm \sqrt{7}$. Since $x > 0$, we have $PA = 2x = 2(\sqrt{7} - 1) \approx 3.29$. A larger minimum distance can be achieved with the second configuration at the right. Here the minimum distance is clearly equal to XU and by the Pythagorean Theorem again, we have $XU = \sqrt{2^2 + 3^2} = \sqrt{13} \approx 3.61$.



3. Find all square numbers of the form $n^2 + n + 43$ with n a nonnegative integer.

SOLUTION. Note that $n^2 + n + 43 > n^2$ because $n \geq 0$. If $n^2 + n + 43$ is a square, it must be the square of a number exceeding n , and thus we can write $n^2 + n + 43 = (n + k)^2$, where k is some positive integer. This yields $n + 43 = k^2 + 2nk$, and thus $43 - k^2 = n(2k - 1) \geq 0$. It follows that k is a positive integer such that $k^2 \leq 43$. In other words, k must be in the set $\{1, 2, 3, 4, 5, 6\}$. Now $n = (43 - k^2)/(2k - 1)$, and this must be an integer. If we check the six possible values for k , we discover that $(43 - k^2)/(2k - 1)$ is an integer precisely when k is 1, 2 or 5. These correspond to

$n = 42$, $n = 13$ and $n = 2$. For these three choices of n , the values of $n^2 + n + 43$ are $1849 = 43^2$, $225 = 15^2$ and $49 = 7^2$. These, therefore, are the only squares of the form $n^2 + n + 43$.

4. Consider the following two-person game. We start with three piles of 10 coins each, with one pile designated as the “hot” pile. On his or her turn, a player can take any positive number of coins from any non-hot pile and put those coins on any other pile. The destination pile then becomes the new hot pile. Note that no move is possible if all 30 coins are in the hot pile. The game is over when some player cannot make a move, and in that case, that player loses. Prove that the second player can always force a win.

SOLUTION. The winning strategy for the second player (whom we will call Two) is to arrange that after each of her moves, the two non-hot piles have equal sizes, and the hot pile is larger than it was before her opponent’s previous move. Assuming for the moment that Two can really do this, we see that on each of his moves, the first player (named One, of course) will be presented with a larger and larger hot pile, and so eventually he must be faced with the situation that all of the coins are in the hot pile. At that point, One cannot move and Two wins.

Suppose that at some point in the game when it is One’s turn, the two non-hot piles each have x coins and the hot pile has y coins, where $y \geq x$. (In particular, this is the situation at the start of the game, when $x = 10 = y$.) If One decides to transfer $t > 0$ coins, he has essentially just two options: he can move t coins from one pile of size x to the hot pile of size y , or he can move t coins from one pile of size x to the other. If he chooses the first option, then Two can respond by moving t coins from the other pile of size x to the hot pile (which now contains $y + t$ coins). What results is two equal non-hot piles of size $x - t$ and a hot pile of size $y + 2t$. Suppose instead that One chooses his second option. Two is then presented with piles of size $x - t$, $x + t$ and y , where the pile of size $x + t$ is hot. Since $y \geq x$, we see that $y > x - t$, and Two can transfer $y - x + t$ coins from the (non-hot) pile of size y to the (hot) pile of size $x + t$ leaving exactly $x - t$ coins in the pile that formerly had y coins. After this move, there are two non-hot piles, each containing $x - t$ coins and a hot pile containing $y + 2t$ coins. Thus Two can counter either of the two options and hence she always win the game, regardless of what One does.

5. Observe that the number $n = 536$ has no repeated digits and that its double, $2n = 1072$ also has no repeated digits and has no digit in common with n . Find the largest integer n such that n and $2n$ have no repeated digits and have no digits in common.

SOLUTION. The largest number with the stated property is $n = 48651$, so that $2n = 97302$. Suppose there were a larger solution $m > n$. Since m has at least five digits and there is a total of at most ten digits available, the number $2m$ cannot have more than five digits. But $2m > 2n$, and so we conclude that $2m$ begins with the digit 9, and hence n begins with 4. The second digit of $2m$ cannot be 9. If it is 8, that forces the second digit of m to be 9, which is impossible because $2m$ begins with 9. Since $2m > 2n$, this forces the second digit of $2m$ to be at least 7, and thus it must be 7. Since $2m = 97 \dots$, we see that $m = 48 \dots$. The third digit of m cannot be 9, 8 or 7 because we have already used these digits, and since $m > n$, the third digit of m cannot be less than 6. It follows that the third digit of m must be 6. Similar reasoning shows that the fourth digit of m is 5. The only digits that remain unused at this point are 0, 1, 2 and 3, and since $m > n$, the final digit of m must be 2 or 3. The final digit of $2m$, therefore, would be 4 or 6, but neither of these is possible because these digits have already been used. We have reached a contradiction, and this shows that there cannot be a solution with $m > n$.