THE COMPACT-OPEN TOPOLOGY: WHAT IS IT REALLY?

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Recall from algebraic topology:

(1) Path: $[0,1] \to X$
(2) Loop: $S^1 \to X$
(3) Homotopy = path of paths: $[0,1] \to C([0,1], X)$, but really $[0,1] \times [0,1] \to X$.
(4) Loop of loops = $S^1 \to C(S^1, X)$, but really a donut: $S^1 \times S^1 \to X$.

Why are these equivalent? For what topology on $C(S^1, X)$? This talk is a streamlined exposition of [EH].

1. Exponential topologies.

Definition.

(1) A topology on $C(X,Y)$ is called:
  - *weak* if $A \times X \xrightarrow{f} Y$ continuous $\Rightarrow A \xrightarrow{\overline{f}} C(X,Y)$ continuous. Weaker than weak $\Rightarrow$ weak.
  - *strong* if $A \xrightarrow{\overline{f}} C(X,Y)$ continuous implies $A \times X \xrightarrow{f} Y$ continuous; equivalently if $C(X,Y) \times X \xrightarrow{ev} Y$ is continuous. Stronger than strong $\Rightarrow$ strong.
  - *exponential* if both: $\{A \to C(X,Y)\} \simeq \{A \times X \to Y\}$. This is unique because weak is weaker than strong.

(2) The *Sierpinski space* $\mathbb{S}$ is $\{0,1\}$ with open $\{1\} \subseteq \mathbb{S}$. The opens sets of X are $\mathcal{O}X = C(X, \mathbb{S})$.
(3) Canonical open map $Y \to \mathbb{S}$ induces $C(X,Y) \to C(X,\mathbb{S})$ and thus T on $C(X,\mathbb{S})$ induces weakest topology so that $C(X,Y) \to C(X,\mathbb{S})$ is continuous: generated by $T(O,V) = \{f \in C(X,Y) \colon f^{-1}(V) \in O\}$ where O is an open set (of open sets of X) in T, and V is an open set of Y.

Proof of the assertions in (1). To see that a topology is strong if and only if evaluation is continuous, notice that $\overline{\text{ev}} \colon C(X,Y) \to C(X,Y)$ is the identity map, so is continuous in any topology on $C(X,Y)$. Thus in a strong topology on $C(X,Y)$, evaluation is continuous. Conversely, note that for any $g \colon A \times \to Y$, we have that $\overline{g} \colon A \to C(X,Y)$ is given by $\text{ev} \circ (\overline{g} \times \text{id}_X)$, which is continuous if ev and $\overline{g}$ are.

To see that topologies weaker than a weak topology are weak, and ones stronger than a strong topologyy are strong, notice that that any continuous $\overline{g} \colon A \to C(X,Y)$ remains continuous if $C(X,Y)$ is made weaker. For the third claim, if $S(X,Y)$ is a strong topology on $C(X,Y)$, then from the previous lemma $\text{ev} \colon S(X,Y) \times X \to Y$ is continuous. If $W(X,Y)$ is a weak topology on $C(X,Y)$, then by definition $\overline{\text{ev}} \colon S(X,Y) \to W(X,Y)$ is continuous. But $\overline{\text{ev}}$ is the identity set-map, hence $W(X,Y)$ is weaker than $S(X,Y)$, as desired. ☐

Proposition. T on $C(X,\mathbb{S}) = \mathcal{O}X$ is weak/strong (so exponential) $\Leftrightarrow$ the induced topology on $C(X,Y)$ is weak/strong (so exponential) for every Y. Such an X is called *exponentiable* (exponentiable X are precisely the topological spaces for which the functor $- \times X$ is part of an adjunction $- \times X \vdash C(X, -)$).

Proof. One direction is clear: simply take $Y = \mathbb{S}$ so that $T(X,\mathbb{S}) = T$. For the other direction, we have two cases: when T is weak and when T is strong.

In the case where T is weak, we are intersted in continuity of $\overline{g} \colon A \to T(X,Y)$, i.e. in realzing the set $\{a \in A \colon \overline{g}(a)^{-1}(V) \in O \subseteq C(X,\mathbb{S})\}$ as an open subset of X for each open $V \subseteq Y$ and each open $O \subseteq C(X,\mathbb{S})$. Since $g \colon A \times X \to Y$ is continuous, we have that $g^{-1}(V) = W \hookrightarrow A \times X$ is open, hence corresponds to a continuous map $w \colon A \times X \to \mathbb{S}$ with continuous (by weakness of $C(X,\mathbb{S})$) curry $\overline{w} \colon A \to C(X,\mathbb{S})$. This curry has the property that $\overline{w}(a) = \overline{g}(a)^{-1}(V)$. The fact that $\overline{w}^{-1}(O) \subseteq A$ is open completes the proof.

In the case where T is strong, it suffices to show $\epsilon_{X,Y} \colon C(X,Y) \times X \to Y$ is continuous presuming $\epsilon_X \subseteq C(X,\mathbb{S}) \times X$ is open ($(O,x) \in \epsilon_X$ if and only if $x \in O$). Taking $V \subseteq Y$ an open neighborhood of $f(x) = \epsilon_{X,Y}(f,x)$, we have $(f^{-1}(V), x) \in \epsilon_X$ by continuity of f. Since ϵ_X is open, there exists an open

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neighborhood $O \times U \subseteq \epsilon_X$ of $(f^{-1}(V), x)$, which means that $(f, x) \in T(O, V) \times U$. All that remains is to show that $\epsilon_{X,Y}(T(O, V) \times U) \subseteq V$, but this follows since $(g, u) \in T(O, V) \times U$ means $(g^{-1}(V), u) \in O \times U$, which being contained in ϵ_X means that $u \in g^{-1}(V)$, i.e. $\epsilon_{X,Y}(g, u) = g(u) \in V$. □

2. Topologies on $C(X, \mathbb{S}) = \mathcal{O}X$.

Proposition. Any topology T on $\mathcal{O}X$ gives rise to a *"funneled by" relation* $\prec_T$ on $\mathcal{O}X$ given by $U \prec_T V$ if there is an open neighborhood of V in $\mathcal{O}X$ consisting of open sets containing U. Such relations $\prec_T$ satisfy:

(1) transitive: $U \prec_T V \prec_T W$ implies $U \prec_T W$
(2) weaker than $\subseteq$: $U \prec_T V$ implies $U \subseteq V$
(3) $\subseteq$-downward closed: $U' \subseteq U \prec V$ implies $U' \prec V$
(4) $\subseteq$-finite coproduct closed: $\emptyset \prec_T V$ for any V, and $U_1, U_2 \prec_T V$ implies $U_1 \cup U_2 \prec_T V$.

Conversely, $\{\text{open } V \subseteq X : U \prec V\}$ are basic opens for a topology T with $\prec_T = \prec$. The topology on $C(X, Y)$ induced by a relation $\prec$ is generated by $T(U, V) = \{f \in C(X, Y) : U \prec f^{-1}(V)\}$, for U and V in $\mathcal{O}X$. Note that different topologies can give rise to the same $\prec_T$, but different $\prec$ give rise to different topologies.

Example.

(1) The *Alexandroff topology* $T_\emptyset$ generated by the relation $\subseteq$. Its open sets are simply upward-closed subsets of $\mathcal{O}X$, with basic open sets $O_U = \{\text{open } V \subseteq X : U \subseteq V\}$. (Notice that reflexivity of $\prec_T$ implies T is stronger than the Alexandroff topology, and $\subseteq$-upward closure of $\prec_T$ ($U \prec_T V \subseteq W$ implies $U \prec_T W$) is equivalent to T being weaker thant the Alexandroff topology.)
(2) The topologies $T_\mathcal{C}$ for $\mathcal{C} \subseteq \mathcal{O}X$ whose open sets are Alexandroff opens O such that if $\mathcal{C}$ is an open cover of an element of O, then it has a finite subcover of some element in O. These are obviously weaker than the Alexandroff topology, and, except for the case $\mathcal{C} = \emptyset$, these are different than the topology generated by the relaton $\prec_{T_\mathcal{C}}$.
(3) The *Scott topology* whose open sets are Alexandroff opens O such that any open cover of an element in O has a subcover of an element in O. Clearly, the Scott topology is the intersection of the topologies $T_\mathcal{C}$ (so weaker than all of them), and is likewise a priori not the same as the topology generated by the relation $\prec_{T_{\text{Scott}}}$. Significantly, the Scott topology is weak.
(4) The *relative compact topology* generated by the relation $U \ll V$ if and only if any open cover of V has a finite subcover of U. This is weaker than the Alexandroff topology, but in general incomparable to the others.

On the other hand, it is easy to see that $\prec_{T_{\text{Scott}}}$ is a weaker relation than $\ll$ since if O is an Alexandroff open in O_U containing V with any cover of an element in O having a finite subcover of an element in O, then certainly any open cover of V has a finite subcover of U, i.e. certainly $U \ll V$.

Proof that the Scott topology is weak. To show that the Scott topology is weak, take $W \subseteq A \times X$ an open set. We wish to show that $\overline{w} : A \to C(X, \mathbb{S})$ is continuous at every point $a \in A$, so fix $a \in A$ and a basic Scott open neigborhood O of $\overline{w}(a) \in C(X, \mathbb{S})$.

Since $W \subseteq A \times X$ is open, for each $x \in \overline{w}(a) \subseteq X$ we have an open neighborhood $U_x \times V_x \subseteq W$ of (a, x). Certainly, $\overline{w}(a) \in O$ is the union of the open neighborhoods V_x, hence the union V for some finite subcollection of those V_x is in O also. We set U to be the intersection of the corresponding finitely many U_x, which is an open neighborhood of a. We now want to show that $\overline{w}(U) \subseteq O$, i.e. that $\overline{w}(u) \in O$ for every $u \in U$. Since we have $V \in O$, it is sufficient to show that $\overline{w}(U) \in O_V \subseteq O$, i.e. that $V \subseteq \overline{w}(u)$ for each u. But for every $v \in V$, we have $(u, v) \in U_x \times V_x \subseteq W$ for some $x \in \overline{w}(a)$, so indeed $v \in \overline{w}(u)$. □

Definition. A topology T is *approximating* if for any open neighborhood V of x, there is an open neighborhood of U of x so that $U \prec_T V$. In particular, every open V is the union of the opens it "funnels".

Lemma.

- *$T_\mathcal{C}$ are all approximating.*
- *A topology T on $C(X, \mathbb{S})$ is strong if and only if it is approximating. Hence, the Scott topology is the strongest weak topology, and X is exponentiable if and only if the Scott topology is approximating.*
- *If $\preceq$ is approximating, and $\prec'$ is approximating and upward $\subseteq$-closed (i.e. weaker than the Alexandroff topology), then for any $U \ll V$, there exists W such that $U \preceq' W \preceq V$.*

Proof. The claim that $T_{\mathcal{C}}$ are approximating is is easy to check as follows. If $x \notin \bigcup \mathcal{C}$, then $\mathcal{C}$ does not cover V, hence O_V is an open Alexandroff open containing V so $V \prec_{T_{\mathcal{C}}} V$. If $x \in \bigcup_{\mathcal{C}}$, then $x \in U$ for some $U \in \mathcal{C}$, and we easily have that O_U is an open Alexandroff open containing V so again $U \prec_{T_{\mathcal{C}}} V$.

For the relationship of being approximating to strength, first let us suppose that T is strong, i.e. that $\epsilon_X \subseteq \mathcal{O}X \times X$ is open. Then V being an open neighbrohood of x means that $(V, x) \in \epsilon_X$, hence the latter point has an open neighborhood $O \times U \subseteq C(X, \mathbb{S}) \times X$. But now $W \in O$ implies $(W, u) \in O \times U$ implies $u \in W$ for every $u \in W$, hence that $U \subseteq W$ and so $W \in O_U$. Consequently, O is an open subset of O_U and $U \prec_T V$ as V is in the interior of O_U.

Conversely, suppose that T is approximating. We wish to show that $\epsilon_X \subseteq C(X, \mathbb{S}) \times X$ is open. Pick $(V, x) \in \epsilon_X$ and consider a neighbrhood U of x such that $U \prec_T V$. Take $O \subseteq C(X, \mathbb{S})$ to be an open neighborhood of V contained in O_U. Then $(V, x) \in O \times U \subseteq \epsilon_X$, as desired.

Note that the relation $\prec$ being approximating means that any open V is the union of the open W such that $W \prec V$. But furthermore, each open W is the union of open W' such that $W' \prec' W$. In particular, we have an open cover of V consisting of all those open sets W' for which the exists an open W such that $W' \prec' W \prec V$. This cover is evidently closed under finite unions (by the upward and finite coproduct closure of the relations), hence $U \ll V$ implies there is a single open W' that covers U, so we have in total $U \subseteq W' \prec' W \prec V$. Since $\prec'$ is downward closed, we have $U \prec' W \prec V$, as desired. □

Theorem. *If the Scott topology is approximating, then $\ll$ is also approximating. If $\ll$ is approximating, then the relative compact topology is weaker than the Scott topology, hence the two coincide and are the exponential topology.*

Proof. For the first claim, taking $\prec'$ to be $\subseteq$ and applying downward closure of $\prec$ gives us that $U \ll V$ implies $U \prec V$, i.e. that $\ll$ is weaker than any approximating relation. In particular, if the Scott topology is approximating, then $\ll$ is also approximating.

For the second claim, if $\ll$ is approximating, then taking $\prec$ and $\prec'$ to be the approximating $\ll$ in the lemma, we obtain that for any $U \ll V$ there exists a W such that $U \ll W \ll V$. Consequently, if $\mathcal{C}$ is an open cover of some V in the Alexandroff open $\{\text{open } V \subseteq X \colon U \ll W\}$, there is a W in that open of which $\mathcal{C}$ has a finite subcover. This shows that the relatively compact topology is weaker than the Scott topology. □

Definition. We say that a topological space X is *core-compact* if the relation $\ll$ on $\mathcal{O}X$ is approximating, i.e. if any open neighborhood V of x has an open neighborhood U of x such that $U \ll V$, that is, any open cover of V has a finite subcover of U. Then the *core-open* topology on $C(X, Y)$ is generated by $T(U, V) = \{f \in C(X, Y) : U \ll f^{-1}(V)\}$

3. **Compact-open topology.**

Proposition. A space X is locally compact if and only if it is core-compact and whenever $U \ll V$, there exists a compact K such that $U \subseteq K \subseteq V$. In particular, for a locally compact X the exponential topology on $C(X, Y)$ is the compact-open topology generated by $T(K, V) = \{f \in C(X, Y) \colon f(K) \subseteq V\}$.

Proof. If X is locally compact, then given a neighborhood an open neighborhood V of x, there is a compact neighborhood K of x, so with $U = \text{int}(K)$ we have $U \subseteq K \subseteq V$, which obviously implies $U \ll V$. Furthermore, given $U \ll V$, for each $x \in V$ choose a compact neighborhood K_x of x contained in V. Then the $\text{int}(K_x)$ are an open cover of V, hence they have a finite subcover of U. It follows that the union of those finitely many K_x's is a compat K such that $U \subseteq K \subseteq V$.

Conversely, if X is core-compact and then any open neighborhood of V of x contains an open neighborhood U of x such that $U \ll V$. Then the compact K such that $U \subseteq K \subseteq V$ is by definition a compact neighborhood of x contained in V, so X is locally compact. □

For the sake of completeness of the exposition, the following is taken from the delightful [GL13], which in particular contains very pretty diagrams on pages 154-155 that summarize the relationships between the various topologies on $C(X, Y)$.

Definition. A *sober space* X is one in which every irreducible closed subset X is the closure of a singleton.

Theorem (Hofmann-Mislove, 8.3.2 in [GL13])**.** *Suppose X is a sober space, and $\mathcal{U} \subseteq \mathcal{O}X$ a Scott open filter, i.e. a Scott open subset (automatically upward closed) which is closed under finite intersections: for any $U_1, U_2 \in \mathcal{U}$, we have $U_1 \cap U_2 \in \mathcal{U}$. Then $Q = \bigcap_{U \in \mathcal{U}} U$ is compact and (trivially) saturated (the intersection of open sets), and $\mathcal{U}$ consists of the open neighborhoods of Q.*

Proof. As remarked, it is trivial that Q is saturated. Further, to show compactness it suffices to show that $\mathcal{U}$ consists of the open neighborhoods of Q since then any open cover $\mathcal{C}$ of Q will be a cover of an element in $\mathcal{U}$, hence will have a finite subcover of an element of $\mathcal{U}$, so a finite subcover of Q.

We proceed to show that there does not exist an open U of Q outside $\mathcal{U}$. Since $\mathcal{U}$ is Scott open, the union of any ascending chain of such opens being in $\mathcal{U}$ will imply a finite union of them was in $\mathcal{U}$. Hence Zorn's lemma magnifies the presence of a single open U of Q outside of $\mathcal{U}$ to a maximal such.

We claim that the complement F of this maximal U is irreducible. Since $\mathcal{U}$ is upward-closed, it contains X, so U is proper and F is non-empty. Next, if F intersects two opens U_1 and U_2, neither of them is in $\mathcal{U}$. Since both $U \cup U_1$ and $U \cup U_2$ are in $\mathcal{U}$ by maximality, we obtain that $U \cup (U_1 \cap U_2)$ is also in $\mathcal{U}$, implying that F meets $U_1 \cap U_2$. But this shows that F is irreducible.

Since X is assumed sober, F is the closure of a singleton $\{x\}$, and so $x \notin Q$ since it is in the complement of U, a neighborhood of Q. But by definition of Q, this means that $x \notin U'$ for some $U' \in \mathcal{U}$, so U' is disjoint from F and is thus in $U \notin \mathcal{U}$, contradicting the upward closure of $\mathcal{U}$. □

Corollary (Theorem 8.3.10 in [GL13])**.** A sober space is exponentiable if and only if it is locally compact.

Proof. Consider a core-compact space X, and take U_0 an open neighborhood of a point x. Core-compactness implies that we can find an open neighborhood of x such that $U_\omega \ll U_0$, and then for any $U_\omega \ll U_i$ we can find an open U_{i+1} such that $U_\omega \ll U_{i+1} \ll U_i$. Thus we obtain $U_\omega \ll \cdots \ll U_{i+1} \ll U_i \ll \cdots \ll U_0$. Consequently, we have a Scott-open $\mathcal{U}$ of open sets V such that $U_n \ll V$ for some n, which is clearly a filter. When X is sober, we get that $K = \bigcap_{V \in \mathcal{U}} V$ is a compact set such that $U_\omega \subseteq K \subseteq U_0$. Thus, K is a compact neighborhood of x as desired. □

References

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[GL13] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology*, volume 22. Cambridge University Press, 2013.