Math 763.
Correspondence between ideals and algebraic sets.

In this note, we list the properties of the operations I and V. Recall that

- For a subset $S \subset k[x_1, \dots, x_n]$, $V(S)$ is the common zero locus of S;
- For a subset $X \subset k^n$, $I(X)$ is the set of all functions vanishing on X.

The following properties are supposed to be obvious; if they are not, treat them as exercises. (And try not to use the Nullstellensatz unless absolutely necessary.)

(1) $I(X) = \bigcap_{a \in X} \mathfrak{m}_a$, where $\mathfrak{m}_a = (x - a_1, \dots, x - a_n)$ for $a = (a_1, \dots, a_n) \in k^n$. (Reformulation of the definition.)
(2) $V(S) = \{a \in k^n : \mathfrak{m}_a \supset S\}$. (Reformulation of the definition.)
(3) $I(X)$ is a radical ideal.
(4) $V(S)$ is an algebraic set (by definition).
(5) If $S_1 \subset S_2$, then $V(S_1) \supset V(S_2)$.
(6) If $X_1 \subset X_2$, then $I(X_1) \supset I(X_2)$.
(7) $I(\emptyset) = k[x_1, \dots, x_n]$, $I(k^n) = 0$.
(8) $V(\emptyset) = V(0) = k^n$, $V((1)) = V(\{1\}) = \emptyset$.
(9) $V(I(X))$ is the smallest algebraic set containing X.
(10) $I(V(S)) = \sqrt{\text{ideal generated by } S}$.
(11) $V(S_1 \cup S_2) = V(S_1) \cap V(S_2)$, more generally, $V(\bigcup_\alpha S_\alpha) = \bigcap_\alpha V(S_\alpha)$.
(12) $I(X_1 \cap X_2) = I(X_1) \cup I(X_2)$, more generally, $I(\bigcap_\alpha X_\alpha) = \bigcap_\alpha I(X_\alpha)$.
(13) $V(S_1 \cdot S_2) = V(S_1) \cup V(S_2)$, where $S_1 \cdot S_2 := \{f_1 f_2 : f_1 \in S_1, f_2 \in S_2\}$.

Now let us restrict the operation V to (not necessarily radical) ideals.

(14) $V(I_1 + I_2) = V(I_1) \cap V(I_2)$;
(15) More generally, $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$;
(16) $V(I_1 \cdot I_2) = V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.

Note that even if $I_1 = \sqrt{I_1}$ and $I_2 = \sqrt{I_2}$, it is not necessarily true that $\sqrt{I_1 + I_1} = (I_1 + I_2)$ or $\sqrt{I_1 \cdot I_2} = (I_1 \cdot I_2)$, but it is true that $\sqrt{I_1 \cap I_2} = I_1 \cap I_2$. We then see that $\sqrt{I_1 \cdot I_2} = I_1 \cap I_2$.

Finally, if we restrict I and V to algebraic subsets of k^n and radical ideals of $k[x_1, \dots, x_n]$ respectively, they become order-reversing bijections.