# ALGEBRA QUALIFYING EXAM, JANUARY 2017

**1.** For this problem (and this problem only) your answer will be graded on correctness alone, and no justification is necessary. Give an example of:

(a) A group $G$ with a normal subgroup $N$ such that $G$ is not a semidirect product $N \rtimes G/N$.
(b) A finite group $G$ that is nilpotent but not abelian.
(c) A group $G$ whose commutator subgroup $[G, G]$ is equal to $G$.
(d) A non-cyclic group $G$ such that all Sylow subgroups of $G$ are cyclic.
(e) A transitive action of $S_3$ on a set $X$ of cardinality greater than 3.

**2.** Let $n > 0$ be an integer. Let $F$ be a field of characteristic 0, let $V$ be a vector space over $F$ of dimension $n$, and let $T : V \to V$ be an invertible $F$-linear map such that $T^{-1} = T$.

Denote by $W$ the vector space of linear transformations from $V$ to $V$ that commute with $T$. Find a formula for $\dim(W)$ in terms of $n$ and the trace of $T$.

**3.** Let $R$ be a commutative ring with unity. Show that a polynomial

$$f(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0 \in R[t]$$

is nilpotent if and only if all of its coefficients $c_0, \ldots, c_n \in R$ are nilpotent.

**4.** This is a question about "biquadratic extensions," in two parts.

(a) Let $F/E$ be a degree-4 Galois extension, where $E$ and $F$ are fields of characteristic different from 2. Show that $\mathrm{Gal}(F/E) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ if and only if there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ and none of $x, y, xy$ are squares in $E$.
(b) Give an example of a field $E$ of characteristic 2 that is not algebraically closed but that has no Galois extension $F/E$ with Galois group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

**5.** Consider the ring $R = \mathbb{C}[x]$.

(a) Describe all simple $R$-modules.
(b) Give an example of an $R$-module that is indecomposable, but not simple. (Recall that a module is *indecomposable* if it cannot be written as a direct sum of non-trivial submodules.)
(c) Consider $R$-modules $M = R/(x^3 + x^2)$ and $N = R/(x^3)$, and take their tensor product over $R$: $M \otimes_R N$. It is an $R$-module, and in particular, a vector space over $\mathbb{C}$. What is its dimension over $\mathbb{C}$?
(d) Let $M$ be any $R$-module such that $\dim_{\mathbb{C}} M < \infty$, and let $N = R/(x^3)$, as before. Show that

$$\dim_{\mathbb{C}}(M \otimes_R N) = \dim_{\mathbb{C}} \mathrm{Hom}_R(N, M).$$

## Solutions

**1.** (a) $G = \mathbb{Z}/4\mathbb{Z}$, $N = 2\mathbb{Z}/4\mathbb{Z}$, or $G = \mathbb{Z}$ and $N = 2\mathbb{Z}$.
(b) The quarternion group $G = \{1, i, j, k, -1, -i, -j, -k\}$.
(c) The alternating group $G = A_5$.
(d) The symmetric group $G = S_3$.
(e) The left action of $S_3$ on itself.

**2.** Since $T^2 = I$, the minimal polynomial of $T$ divides $x^2 - 1 = (x-1)(x+1)$. Therefore, the minimal polynomial of $T$ has no repeated roots. Hence $T$ is diagonalizable, with all eigenvalues among $1$, $-1$. Let $V_\pm := \ker(T \mp I)$ be the eigenspaces of $T$ for the eigenvalues $\pm 1$. Then $V = V_+ \oplus V_-$. We see that

$$n = \dim(V) = \dim(V_+) + \dim(V_-)$$
$$t = \operatorname{tr}(T) = \dim(V_+) - \dim(V_-),$$

and therefore

$$\dim(V_\pm) = \frac{n \pm t}{2}.$$

Finally, an $F$-linear map $S : V \to V$ commutes with $T$ if and only if $S(V_+) \subset V_+$ and $S(V_-) \subset V_-$; therefore, the dimension of the space of such maps is

$$\dim \operatorname{Hom}(V_+, V_+) + \dim \operatorname{Hom}(V_-, V_-) = \dim(V_+)^2 + \dim(V_-)^2 = \frac{n^2 + t^2}{2}.$$

**3.** Recall that the sum of nilpotent elements is nilpotent (which easily follows from the binomial formula); this implies the 'if' direction. For the 'only if', there are (at least) two approaches:

*Direct:* Suppose $f$ is nilpotent. Looking at the constant term of $f^N$, we see that $c_0$ must be nilpotent. This implies that $(f - c_0)$ is nilpotent. Now looking at the lowest coefficient of $(f - c_0)^N$, we conclude that $c_1$ is nilpotent, and so on.

*Less direct:* Suppose $f$ is nilpotent. Since the nilradical is the intersection of all the prime ideals, we need to show that $c_i \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset R$. This is equivalent to showing that $(f \mod \mathfrak{p}) = 0 \in (R/\mathfrak{p})[t]$. However, $(f \mod \mathfrak{p})$ is a nilpotent polynomial over a domain, which clearly implies it must be zero.

**4.** (a) Note first that the quadratic formula implies that any quadratic extension of $E$ is of the form $E(\sqrt{x})$ for $x \notin E^2$; in particular, any quadratic extension is Galois. Moreover, for $x, y \notin E^2$, we see that $E(\sqrt{x}) = E(\sqrt{y})$ if and only if $xy \in E^2$ (which can be seen from looking at the action of the Galois group, or from squaring the expression $\sqrt{y} = a + b\sqrt{x}$).

Now $(a)$ follows from the observation that $\operatorname{Gal}(F/E) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ if and only if $F$ is the composite of two distinct quadratic extensions of $E$. (Of course, there are other proofs.)

(b) The easiest example is probably the Galois field $\mathbb{Z}/2\mathbb{Z}$: the Galois group of any of its finite extension is cyclic.

**5.** (a) The simple modules are of the form $R/\mathfrak{m}$ for maximal ideals $\mathfrak{m}$; since $R$ is the polynomial ring, we see that the maximal modules are of the form $R/(x - a)$ for $a \in \mathbb{C}$.

(b) $R/(x^2)$ has dimension 2 over $\mathbb{C}$, therefore it cannot be simple. However, it is indecomposable: otherwise it would be a direct sum of two simple modules, which contradicts the fact that it has non-trivial nilpotents. (For fancier examples, we could take $R$-modules $\mathbb{C}[x]$ or $\mathbb{C}(x)$.)

(c) Tensoring the exact sequence

$$R \xrightarrow{x^3} R \to N \to 0 \tag{1}$$

by $M$, we obtain the sequence

$$M \xrightarrow{x^3} M \to M \otimes_R N \to 0,$$

so that $M \otimes_R N = \operatorname{coker}(x^3 : M \to M)$. In particular, for $M = R/(x^3 + x^2)$, we get

$$M \otimes_R N = R/(x^3 + x^2, x^3) = R/(x^2),$$

so that $\dim_{\mathbb{C}} M \otimes_R N = 2$.

(d) From the previous part,

$$\dim_{\mathbb{C}} M \otimes_R N = \dim \operatorname{coker}(x^3 : M \to M) = \dim(M) - \operatorname{rk}(x^3 : M \to M).$$

Applying $\operatorname{Hom}_R(-, M)$ to (**??**), we obtain the sequence

$$0 \to \operatorname{Hom}_R(N, M) \to M \xrightarrow{x^3} M,$$

so that $\operatorname{Hom}_R(N, M) = \ker(x^3 : M \to M)$, and

$$\dim_{\mathbb{C}} \operatorname{Hom}_R(N, M) = \dim \ker(x^3 : M \to M) = \dim(M) - \operatorname{rk}(x^3 : M \to M)$$

as well.

(The solution is stated in terms of right/left exactness of the functors, but it can be easily reformulated in more explicit terms.)