Polynomials

1. \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \ a_n \neq 0 \)

   If \( a_n = 1 \), it is called monic.

   The set of polynomials (depending on coefficient ring) is denoted \( \mathbb{C}[x], \mathbb{R}[x], \mathbb{Z}[x], \mathbb{Q}[x], \mathbb{F}_p[x] \), etc.

   If \( P \in \mathbb{C}[x] \), then it has \( n \) complex roots (counting the multiplicity).

   If coefficients are real, then \( n \) complex roots must occur in conjugate pairs.

Example: There is a polynomial \( P \) of degree 7 with integer coefficients. It is known that it is equal to \( \pm 1 \) in 7 integer points. Prove that \( P \) cannot be factorized into the product of two polynomials with integer coefficients, with degree \( \geq 1 \).

2. \( P(x) = Q(x) \cdot R(x) \)

   At least one polynomial of degree \( \leq 3 \), let it \( Q \).

   \( Q = \pm 1 \) in 7 integer points, so at least four \( 1 \) or at least four \(-1 \), which is impossible since \( Q = \pm 1 \) has degree \( \leq 3 \) and so cannot have 4 roots.

Viète’s Relations

\[
\begin{align*}
    x_1 + x_2 + \ldots + x_n &= -\frac{a_{n-1}}{a_n} \\
    x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n &= \frac{a_{n-2}}{a_n} \\
    x_1 x_2 \ldots x_n &= (-1)^n \frac{a_0}{a_n}
\end{align*}
\]

Example: \( x^4 + 3x^3 + 11x^2 + 9x + A \) has roots \( a, b, c, d \) such that \( ab = cd \). Find \( A \)

\[
\begin{align*}
    a + b + c + d &= -3 \\
    ab + ac + ad + bc + bd + cd &= a b (c + d) + (a + b) cd = ab(a + b + c + d) \Rightarrow a b = 3 \\
    \Rightarrow A &= ab \cdot cd = (ab)^2 = 9
\end{align*}
\]
Ex. \( x + y + z = 0 \)  
Prove  
\[
\frac{x^2 + y^2 + z^2}{2} + \frac{x^5 + y^5 + z^5}{5} = \frac{x^7 + y^7 + z^7}{7}
\]

Consider \( t^3 + p \frac{t^2}{2} + q \) with roots \( x, y, z \).

Then \( x^3 = -px - q \).

So  
\[
x^3 + y^3 + z^3 = (-px - q) + (-py - q) + (-pz - q) = -p(x + y + z) - 3q = -3q
\]
\[
x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + xz + yz) = -2p
\]
\[
x^4 = -px^2 - qx
\]
\[
x^4 + y^4 + z^4 = -p(x^2 + y^2 + z^2) - 3q = -3q(x + y + z) = 2p^2
\]
\[
x^5 + y^5 + z^5 = -p(x^3 + y^3 + z^3) - 5q = 5pq
\]
\[
x^7 + y^7 + z^7 = -p(x^5 + y^5 + z^5) - 7q = -7pq
\]

So we get  
\[
\frac{-2p}{2} \cdot \frac{5pq}{5} = \frac{-7pq}{7}
\]

which is true.

3) Derivative: if \( p \equiv 0 \) \( (x-x_0)^r \) \( (x-x_0)^s \)

- \( p(x) = \frac{1}{x-x_1} + \ldots + \frac{1}{x-x_n} \)  
  - if \( p \) has a double root a, then \( p'(a) = 0 \)
  - and if \( p(a) = p'(a) = 0 \) \( \Rightarrow a \) has mult. \( \geq 2 \)

- \( p'(x) \) has root between any two roots of \( p(x) \) (real case)

Ex. Find all points \( x \) \( p(x) \) is a multiple of \( p''(x) \)

If \( a \) is a root of \( p''(x) \), it should be a root of \( p(x) \)

Ex. Prove that \( p(x) \) is a multiple of \( p'(x) \) iff \( p(x) = a(x-x_0)^n \)

\( n p(x) = a(x-x_0) p'(x) \) (compare top coeff.)

Let \( p(x) = (x-x_0)^n Q(x) \) \( Q(x_0) \neq 0 \)
\[
n p(x) = a(x-x_0) \cdot (k(x-x_0)^n + (x-x_0)^{n-1} Q') \Rightarrow n Q(x) = k Q(x) + (x-x_0) Q'(x)
\]
\[
n (x-x_0) Q(x)
\]

Substitute \( x = x_0 \) and get \( n Q(x_0) = k Q(x_0) \Rightarrow n = k \) (since \( Q(x_0) \neq 0 \))

Ex. Is it possible that for each \( a \) \( p(x) = a \) has many number of \( x \)?

Take all horizontal tangent lines and lines between
polynomials has all properties of continuous functions

Ex. \( Q(x) = x \) has no solution. Prove that \( Q(Q(x)) - x \) has no solution.

\( Q(x) - x \) has no roots, so \( Q(x) - x > 0 \) or \( Q(x) - x < 0 \) for all \( x \).

Assume
\[ Q(x) > x \]
Then \( Q(Q(x)) > Q(x) > x \).

Ex. \( P(x) = a_n x^n + \ldots + a_0 \), \( a_n \neq 0 \) has at least one real root.
Prove that one can erase all \( a_i x^i \) one by one in such a way that all intermediate polynomials have at least one real root.

We have \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \).

\( a_n \geq 0 \). If \( n \) is odd, we prove one by one everything except \( a_n x^n + a_0 \), and then \( a_n x^n \) (all the time power is odd \( \Rightarrow \) ok).

So \( n \) is even. Same thing if \( a_n < 0 \), so \( n \) is even and \( a_n > 0 \). Then erasure \( a_n x^n \).

\( Q = a_n x^n + \ldots + a_0 \).
If \( a_n = 0 \), then \( Q(a) = a_{n-1} a_n x^{n-1} < 0 \), \( Q(0) = a_0 > 0 \)
\( \Rightarrow \) there is a root.

Ex. Lagrange interpolation formula

\[ P(x) = a_1 \frac{(x-x_2) \ldots (x-x_{n+1})}{(x_1-x_2) \ldots (x_1-x_{n+1})} + a_2 \frac{(x-x_1)(x-x_3) \ldots (x-x_{n+1})}{(x_2-x_1)(x_2-x_3) \ldots (x_2-x_{n+1})} + \ldots \]

Ex. the polynomial \( P \) has rational values in all rational numbers.
Prove that it has rational coefficients.