Theoretical Introduction to Generating functions

Here are basic recurrence equations that you can solve:

1. Let $a_n$ be a sequence given by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for $n \geq 1$. Find the general term of the sequence $a_n$.

2. Find the general term of the sequence given recurrently by
   
   $a_{n+1} = 2a_n + n$, \quad (n \geq 0), \quad a_0 = 1.

3. $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Find the general term of the sequence.

4. Let the sequence be given by $a_0 = 0$, $a_1 = 2$, and for $n \leq 0$:
   
   $a_{n+2} = -4a_{n+1} - 8a_n$.

   Find the general term of the sequence.

5. Find the general term of the sequence $x_n$ given by
   
   $x_0 = x_1 = 0$, \quad $x_{n+2} - 6x_{n+1} + 9x_n = 2^n + n$ \quad for $n \geq 0$.

6. Let $f_1 = 1$, $f_{2n} = f_n$, and $f_{2n+1} = f_n + f_{n+1}$. Find the general term of the sequence.

7. Evaluate the sum
   
   $\sum_k \binom{k}{n-k}$.

8. Evaluate the sum
   
   $\sum_{k=m}^{n} (-1)^k \binom{n}{k} \binom{k}{m}$.

9. Evaluate the sum
   
   $\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$.
10. Evaluate
\[ \sum_k \binom{n}{k} x^k. \]

11. Determine the elements of the sequence:
\[ f(m) = \sum_k \binom{n}{k} \binom{n-k}{m-k} y^k. \]

12. Prove that
\[ \sum_{k=0}^{n} \binom{2n}{2k} \binom{2k}{k} 2^{2n-2k} = \binom{4n}{2n}. \]

The following problem is slightly harder because the standard idea of snake oil doesn’t lead to a solution.

13. For given \( n \) and \( p \) evaluate
\[ \sum_k \binom{2n+1}{2p+2k+1} \binom{p+k}{k}. \]

14. Prove that for the sequence of Fibonacci numbers we have
\[ F_0 + F_1 + \cdots + F_n = F_{n+2} + 1. \]

15. Given a positive integer \( n \), let \( A \) denote the number of ways in which \( n \) can be partitioned as a sum of odd integers. Let \( B \) be the number of ways in which \( n \) can be partitioned as a sum of different integers. Prove that \( A = B \).

3. Find the number of permutations without fixed points of the set
\[ \{1, 2, \ldots, n\} \]

16. Let \( n \in \mathbb{N} \) and assume that
\[ x + 2y = n \quad \text{has } R_1 \text{ solutions in } \mathbb{N}_0^2 \]
\[ 2x + 3y = n - 1 \quad \text{has } R_2 \text{ solutions in } \mathbb{N}_0^2: \]
\[ nx + (n+1)y = 1 \quad \text{has } R_n \text{ solutions in } \mathbb{N}_0^2 \]
\[ (n+1)x + (n+2)y = 0 \quad \text{has } R_{n+1} \text{ solutions in } \mathbb{N}_0^2 \]
Prove \( \sum_k R_k = n + 1 \).
17. A polynomial \( f(x_1, x_2, \ldots, x_n) \) is called a symmetric if each permutation \( \sigma \in S_n \) we have \( f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n) \). We will consider several classes of symmetric polynomials. The first class consists of the polynomials of the form:
\[
\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}
\]
for \( 1 \leq k \leq n, \sigma_0 = 1, \sigma_k = 0 \) for \( k > n \). Another class of symmetric polynomials are the polynomials of the form
\[
p_k(x_1, \ldots, x_n) = \sum_{i_1 + \cdots + i_n = k} x_1^{i_1} \cdots x_n^{i_n}, \quad \text{where } i_1, \ldots, i_n \in \mathbb{N}_0.
\]
The third class consists of the polynomials of the form:
\[
s_k(x_1, \ldots, x_n) = x_1^k + \cdots + x_n^k.
\]
Prove the following relations between the polynomials introduced above:
\[
\sum_{r=0}^{n} (-1)^r \sigma_r p_{n-r} = 0, \quad np_n = \sum_{r=1}^{n} s_r p_{n-r}, \quad \text{and } n\sigma_n = \sum_{r=1}^{n} (-1)^{r-1} s_r \sigma_{n-r}.
\]

18. Prove that there is a unique way to partition the set of natural numbers in two sets \( A \) and \( B \) such that: For every non-negative integer \( n \) (including 0) the number of ways in which \( n \) can be written as \( a_1 + a_2, a_1, a_2 \in A, a_1 \neq a_2 \) is at least 1 and is equal to the number of ways in which it can be represented as \( b_1 + b_2, b_1, b_2 \in B, b_1 \neq b_2 \).

19. Prove that in the contemporary calendar the 13th in a month is most likely to be Friday. Remark: The contemporary calendar has a period of 400 years. Every fourth year has 366 days except those divisible by 100 and not by 400.

20. Let \( a \) and \( b \) be positive integers. For a non-negative integer \( n \) let \( s(n) \) be the number of non-negative integer solutions to the equation
\[
ax + by = n
\]
Prove that the generating function of the sequence \( s(n) \) is
\[
f(x) = \frac{1}{(1-x^a)(1-x^b)}
\]

21. Prove that the number of ways of writing \( n \) as a sum of distinct positive integers is equal to the number of ways of writing \( n \) as a sum of odd positive integers. Note: This property is usually phrased as follows: Prove that the number of partitions of \( n \) into distinct parts is equal to the number of partitions of \( n \) into odd parts.

Generating functions are powerful tools for solving a number of problems mostly in combinatorics, but can be useful in other branches of mathematics as well.