Generating functions and telescoping series.

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Generating functions:
Encode the sequence \(a_0, a_1, \ldots\) via
\[
f(x) = \sum_{k=0}^{\infty} a_k x^k.
\]

Telescoping series:
\[
(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0.
\]

(Source: NWU Putnam preparation.)

1. Show that
\[
\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}.
\]

2. (Follow-up) Find a closed formula for
\[
\sum_{k=0}^{n} k^2 \binom{n}{k}.
\]

3. Let \(F_n\) be the Fibonacci sequence: \(F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}\).
   Find
\[
\sum_{k=0}^{\infty} \frac{F_k}{2^k}.
\]

4. (Follow-up) Find
\[
\sum_{k=0}^{\infty} k \cdot F_k.
\]

5. How many different sequences are there that satisfy all of the following conditions:
   - The terms of the sequences are the digits 0–9.
   - The length of the sequences is 6 (e.g. 061030).
   - Repetitions are allowed.
   - The sum of the digits is exactly 10 (e.g. 111322).

6. Prove that
\[
\frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \cdots + \frac{1}{\sqrt{99} + \sqrt{100}} = 9.
\]

7. Let \(N\) be a positive integer and let \(S_N\) be the sum
\[
S_N = \frac{1}{2} \sum_{k=1}^{N^2} \frac{1}{\sqrt{k}}.
\]
   Find \([S_N]\), where \([x]\) is the largest integer less than or equal to \(x\).

8. Find a closed form for
\[
\sum_{k=1}^{n} k \cdot k!.
\]
9. (Putnam 1984) Express
\[ \sum_{k=1}^{\infty} \frac{6^k}{(3k+1 - 2k+1)(3k - 2k)} \]
as a rational number.

10. (Putnam 1977) Evaluate the infinite product
\[ \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} \]

11. Evaluate the infinite series:
\[ \sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1} \]