GAMES (10/15/14)

Warm-up

1. Two players take stones from a pile of n (the turns alternate). Each player is allowed to take any number of stones between 1 and 3. The goal is to take the last stone. Which player wins? (The answer depends on n.)

2. Two players take turns putting dominoes on a round table without overlap. The goal is to place the last domino. Show that the first player can always win.

3. Two players take turns writing positive integers on the board. The first number must be between 1 and 10; the number written on each turn must be between $2N$ and $10N$, inclusive, where N is the number written on the previous turn. The goal is to be the first to write 10^{10}. Who wins?

Easier problems

4. Two players take stones from a pile of n (the turns alternate). The number number of stones taken on each turn must be a power of 2. The goal is to take the last stone. Which player wins? (The answer depends on n.)

5. A and B alternately place signs $+$, $-$, $\times$ in empty spaces in the sequence 1 2 3 ... 100. Show that A can make result (a) even, (b) odd.

Actual competition problems

6. (2002-B2) Consider a polyhedron such that (1) each vertex have exactly three edges emerging from it and (2) there are at least four faces. Two players take turns signing their names on (previously unsigned) faces. The goal is to sign three faces that share a common vertex.

Show that the first player can always win.

7. (2008-A3) (One-player game) Start with a finite sequence $a_1, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by their GCD and LCM, respectively. Prove that if this process is repeated, it must eventually stop, and that the final sequence does not depend on the choices made. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.