Putnam Club Week Four – 9 October 2012

Calculus

1. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0? (Putnam 2011)

2. Evaluate $\displaystyle\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive. (Putnam 1989)

3. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s. (Putnam 1998)

4. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

has exactly n distinct real roots? (Putnam 1990)

5. Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\, dt$. Evaluate

$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

(Putnam 2008)

6. Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\, dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)\, dx \right| \leq \frac{1}{8} \max_{0\leq x\leq 1} |f'(x)|.$$

(Putnam 2007)

7. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.) (Putnam 2008)