Last time we started working on the following 5 problems. For some of them we
found solutions, but for some we did not. Let’s pick up from where we left! I am
adding more problems to this working sheet, hoping they will be a good exercise for
the coming up exam!

1. Find all differentiable functions \( f : (0, \infty) \to (0, \infty) \) for which there is a positive
real number \( a \) such that
\[
\frac{d}{dx} \left( a \frac{f(x)}{x} \right) = x f(x)
\]
for all \( x > 0 \).

**Proof.** Substitute \( a/x \) for \( x \) in the given equation:
\[
f'(x) = \frac{a}{xf(a/x)}.
\]
Differentiate:
\[
f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.
\]
Now substitute to eliminate evaluations at \( a/x \):
\[
f''(x) = -\frac{f'(x)}{x} + \frac{f''(x)}{f(x)}.
\]
Clear denominators:
\[
x f(x) f''(x) + f(x) f'(x) = x f'(x)^2.
\]
Divide through by \( f(x)^2 \) and rearrange:
\[
0 = \frac{f'(x)}{f(x)} + \frac{x f''(x)}{f(x)} - \frac{x f'(x)^2}{f(x)^2}.
\]
The right side is the derivative of \( x f'(x)/f(x) \), so that quantity is constant. That
is, for some \( d \),
\[
\frac{f'(x)}{f(x)} = \frac{d}{x}.
\]
Integrating yields \( f(x) = cx^d \), as desired.

2. Suppose that \( f \) is a function on the interval \([1, 3]\) such that \(-1 \leq f(x) \leq 1\) for all
\( x \) and \( \int_1^3 f(x) \, dx = 0 \). How large can \( \int_1^3 \frac{f(x)}{x} \, dx \) be?

**Solution:** In all solutions, we assume that the function \( f \) is integrable.
Proof. Let \( g(x) \) be 1 for \( 1 \leq x \leq 2 \) and \(-1\) for \( 2 < x \leq 3 \), and define \( h(x) = g(x) - f(x) \). Then \( \int_1^3 h(x) \, dx = 0 \) and \( h(x) \geq 0 \) for \( 1 \leq x \leq 2 \), \( h(x) \leq 0 \) for \( 2 < x \leq 3 \). Now

\[
\int_1^3 \frac{h(x)}{x} \, dx = \int_1^2 \frac{|h(x)|}{x} \, dx - \int_2^3 \frac{|h(x)|}{x} \, dx
\geq \int_1^2 \frac{|h(x)|}{2} \, dx - \int_2^3 \frac{|h(x)|}{2} \, dx = 0,
\]

and thus \( \int_1^3 \frac{f(x)}{x} \, dx \leq \int_1^3 \frac{g(x)}{x} \, dx = 2 \log 2 - \log 3 = \log \frac{4}{3} \). Since \( g(x) \) achieves the upper bound, the answer is \( \log \frac{4}{3} \).

3. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function with the property that

\[
\int_0^1 f(x) \, dx = \frac{\pi}{4}.
\]

Prove that there exists \( x_0 \in (0, 1) \) such that

\[
\frac{1}{1 + x_0} < f(x_0) < \frac{1}{2x_0}.
\]

Proof. to come! 

4. Let \( f : [0, \infty) \to \mathbb{R} \) be a strictly decreasing continuous function such that \( \lim_{x \to \infty} f(x) = 0 \). Prove that

\[
\int_0^\infty \frac{f(x) - f(x + 1)}{f(x)} \, dx
\]

diverges.

5. Let \( f : [0, 1] \to \mathbb{R} \) be a function for which there exists a constant \( K > 0 \) such that

\[
|f(x) - f(y)| \leq K |x - y|
\]

for all \( x, y \in [0, 1] \). Suppose also that for each rational number \( r \in [0, 1] \), there exist integers \( a \) and \( b \) such that \( f(r) = a + br \). Prove that there exist finitely many intervals \( I_1, \ldots, I_n \) such that \( f \) is a linear function on each \( I_i \) and \( [0, 1] = \bigcup_{i=1}^n I_i \).

6. Let \( f : (1, \infty) \to \mathbb{R} \) be a differentiable function such that

\[
f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)}
\]

for all \( x > 1 \).

Prove that \( \lim_{x \to \infty} f(x) = \infty \).

7. Find all positive integers \( n < 10^{100} \) for which simultaneously \( n \) divides \( 2^n \), \( n - 1 \) divides \( 2^n - 1 \), and \( n - 2 \) divides \( 2^n - 2 \).
8. Given a real number $a$, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some $n$, then the sequence is periodic.

9. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{1}{4} \left( \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right)^2 > 0$$

everywhere. Prove that $f$ is one-to-one.