

Combinatorics

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Euler's formula for planar graphs

First, we recall that a planar graph is a graph embedded in the plane in such a way that edges do not cross. A planar graph divides the plane into regions, each of which is called a face.

Theorem. *Given a connected planar graph denote by V the number of vertices, by E the number of edges, and by F the number of faces (including the infinite face). Then*

$$V - E + F = 2.$$

The elementary proof of Euler's theorem uses induction on the number of edges. However, the theorem also has a deeper interpretation using topology, more specifically, Euler characteristics. The topological approach also applies to graphs embedded in a more general surface.

1. Is it possible to draw paths connecting each pair of 5 houses such that no two paths intersect?
2. Three conflicting neighbors have three common wells. Can one draw nine paths connecting each of the neighbors to each of the wells such that no two paths intersect?

The above two results are the key ingredients in the characterization of all planar graphs. In fact, the Kuratowski's theorem says the following:

A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$.

3. (Putnam and beyond, 850) Consider a polyhedron with at least five faces such that exactly three edges emerge from each vertex. Two players play the following game: the players sign their names alternately on precisely one face that has not been previously signed. The winner is the player who succeeds in signing the name on three faces that share a common vertex. Assuming optimal play, prove that the player who starts the game always wins.
4. (Putnam and beyond, 853) Consider a convex polyhedron whose faces are triangles and whose edges are oriented. A singularity is a face whose edges form a cycle, a vertex that belongs only to incoming edges, or a vertex that belongs only to outgoing edges. Show that the polyhedron has at least two singularities.

Set theory

Example. *Given 2^{n-1} subsets of a set with n elements with the property that any three have nonempty intersection, prove that the intersection of all the sets is nonempty.*

1. Let X be a subset of $\{1, 2, 3, \dots, 2n\}$ with $n+1$ elements. Show that we can find $a, b \in X$ with a dividing b .

2. (Putnam 1995, A1) Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S , then so is ab). Let T and U be disjoint subsets of S whose union is S . Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U , show that at least one of the two subsets T, U is closed under multiplication.
3. (Putnam 1964, B2) Let S be a finite set, and suppose that a collection F of subsets of S has the property that any two members of F have at least one element in common, but F cannot be extended (while keeping this property). Prove that F contains just half of the subsets of S .
4. (Putnam 1996, A4) Let S be a set of ordered triples (a, b, c) of distinct elements of a finite set A . Suppose that
 - (a) $(a, b, c) \in S$ if and only if $(b, c, a) \in S$;
 - (b) $(a, b, c) \in S$ if and only if $(c, b, a) \notin S$ (for a, b, c distinct);
 - (c) (a, b, c) and (c, d, a) are both in S if and only if (b, c, d) and (d, a, b) are both in S .
 Prove that there exists a one-to-one function g from A to \mathbb{R} such that $g(a) < g(b) < g(c)$ implies $(a, b, c) \in S$.
5. (Putnam and beyond, 827) In a society of n people, any two persons who do not know each other have exactly two common acquaintances, and any two persons who know each other don't have other common acquaintances. Prove that in this society every person has the same number of acquaintances.

Geometric combinatorics

1. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = \frac{1}{\sqrt{2}}$. Is this result true for a smaller k ?
2. Show that if the points of the plane are colored black or white, then there exists an equilateral triangle whose vertices are colored by the same color.
3. (USA Mathematical Talent Search, 4/1/18) Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color.
4. (Putnam and beyond, 846) Given a set M of $n \geq 3$ points in the plane such that any three points in M can be covered by a disk of radius 1, prove that the entire set M can be covered by a disk of radius 1.
5. (Putnam 2010, B2) Given that A, B , and C are noncollinear points in the plane with integer coordinates such that the distances AB , AC , and BC are integers, what is the smallest possible value of AB ?
6. (Erdős–Anning Theorem) Is it possible to place infinitely many points in the plane in such a way that all pairwise distances have integer values and points are noncollinear?